Why ~~the~~ curvature and torsion?

In many applications, we would like to think of curves related by a translation and rotation as the same.

These curves definitely do not have the same parametrizations. However if the curves are framed, we can relate the framings.

Recall our definition:

A framing of $\vec{\alpha}(s)$ is a map $F:\mathbb{R}\to SO(3)$ so that $F(s) = \begin{bmatrix} \uparrow & \uparrow & \uparrow \\ \vec{\alpha}' & F_1 & F_2 \\ \downarrow & \downarrow & \downarrow \end{bmatrix}$.

and our theorem:

Given $F:\mathbb{R}\to SO(3)$, we have $F'(s) = F(s)\cdot S(s)$ where the 3×3 matrix $S(s)$ is always skew-symmetric[1].

If F is the Frenet frame, then

$$S(s) = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}$$

---

1. That is $S^T = -S$.

Theorem. If $\vec{\alpha}(t)$ is framed by $F: \mathbb{R} \to SO(3)$, $A \in SO(3)$ is an orthogonal matrix with determinant one[1], and $\vec{v}$ is any vector, then $A\vec{\alpha}(t) + \vec{v}$ is framed by $AF: \mathbb{R} \to SO(3)$.

Further,

$$F' = FS \implies (AF)' = AFS$$

so the matrix S <u>is the same</u> for $\vec{\alpha}(t)$ and $A\vec{\alpha}(t) + \vec{v}$.

Corollary. If $\vec{\alpha}(t)$ is a parametrized curve with curvature and torsion $\kappa(t)$ and $\tau(t)$, then for any $A \in SO(3)$ and $\vec{v}$, $A\vec{\alpha}(t) + \vec{v}$ has the same curvature and torsion.

---

1 That is, a rotation!

In fact, we can prove an even more surprising theorem:

Theorem. If two parametrized space curves $\vec{\alpha}(t)$ and $\vec{\beta}(t)$ have $\vec{\alpha}(0) = \vec{\beta}(0)$ and $\vec{\alpha}'(0) = \vec{\beta}'(0)$ and the same curvature and torsion $\kappa(t)$ and $\tau(t)$, then $\vec{\alpha}(t) = \vec{\beta}(t)$.

This means that (in principle)

"Every property of $\vec{\alpha}$ which doesn't change when $\vec{\alpha}$ is rotated or translated can be expressed in terms of κ and τ."

Proposition. A space curve is a line $\Leftrightarrow$ its curvature $\kappa(s) \equiv 0$.

Proof. ($\Rightarrow$) Since $\gamma(s) = s\vec{v} + \vec{c}$, $T(s) = \vec{v}$, $T'(s) = 0$, and $\kappa \equiv 0$.

($\Leftarrow$) Since $\kappa \equiv 0$, $T'(s) = 0$, and $T(s) = \vec{v}$ for some fixed $\vec{v}$. But then $\gamma'(s) = \vec{v}$. and integrating yields $\gamma(s) = \vec{v}s + \vec{c}$. $\square$

It's a little harder to prove the "lock-on theorem".

Proposition. If all tangent lines of a curve $\gamma(s)$ pass through $\vec{0}$, then γ is a line through $\vec{0}$.

Proof. By hypothesis, $\exists$ some scalar function $\lambda(s)$ so $\gamma(s) + \lambda(s) T(s) = 0$.

or

$$\gamma(s) = -\lambda(s)\,T(s)$$

Differentiating,

$$\gamma'(s) = -\lambda'(s)\,T(s) + \lambda(s)\,\kappa(s)\,N(s),$$

or

$$T(s) = -\lambda'(s)\,T(s) + \lambda(s)\,\kappa(s)\,N(s).$$

Here's a neat trick. We can rewrite this as

$$(1+\lambda'(s))\,T(s) = \lambda(s)\,\kappa(s)\,N(s).$$

But since $T(s)$ and $N(s)$ are orthogonal, this must mean that

$$1+\lambda'(s) = 0 \quad \text{and} \quad \lambda(s)\,\kappa(s) = 0.$$

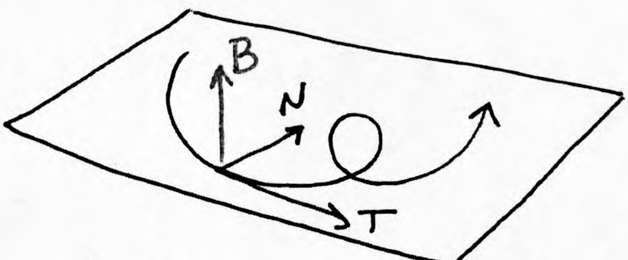
The left equation $\Rightarrow$ $\lambda' \equiv -1$, so $\lambda(s) = c - s$ for some c. Then the right equation $\Rightarrow$ $\kappa \equiv 0$. Thus γ is a line which passes through the origin when $s = c$. $\square$

We now prove something harder.

Proposition. A space curve γ with nonvanishing κ is planar $\Leftrightarrow \tau \equiv 0$.

Wlog, we can assume $\gamma(0) = \vec{0}$ and that γ is parametrized by arclength.

Proof ($\Rightarrow$) If γ is contained in a plane P, at each s, $T(s), N(s)$ are in P. Thus, $T(s) \times N(s)$ is the normal vector to P.



Since this normal is constant

$$B'(s) = -\tau(s) N(s) = 0,$$

and torsion must be zero.

($\Leftarrow$) If $\tau(s) \equiv 0$, $B(s)$ is a constant B_0. Consider $f(s) = \langle \gamma(s), B_0 \rangle$. At $s=0$, $f(0)=0$. But

$$f'(s) = \langle T(s), B_0 \rangle = \langle T(s), B(s) \rangle = 0,$$

so $f(s) \equiv 0$ and $\gamma(s)$ is in the plane normal to B_0. $\square$

Let's review:

$\kappa = 0 \;\Rightarrow\;$ straight line

$\tau = 0 \;\Rightarrow\;$ planar

$\kappa = c_1, \tau = c_2 \;\Rightarrow\;$ helix (homework).

This seems to imply that fixing "one or both of" κ and τ makes the curve very special. This intuition is strengthened by

Proposition. $\kappa(s) \neq 0$ and All tangent vectors of $\gamma(s)$ make a constant angle with some fixed $\vec{v}$ $\Leftrightarrow$ τ/κ is a constant.

A curve like this is called a <u>generalized helix</u>.

Proof. ($\Rightarrow$) We know $\langle T(s), \vec{v} \rangle = \cos\theta =$ constant, so (differentiating),

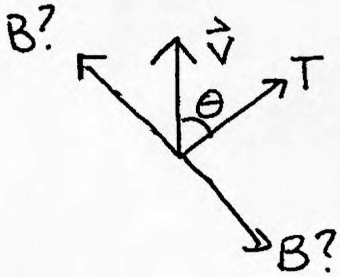
$$\langle \kappa(s) N(s), \vec{v} \rangle = 0$$

Since $\kappa(s) \neq 0$, this implies $\langle N(s), \vec{v} \rangle = 0$.

Now, differentiating again,

$$\langle -\kappa(s)\, T(s) + \tau(s)\, B(s), \vec{v} \rangle = 0$$

Since $\langle N(s), \vec{v} \rangle = 0$, $\vec{v}$ is in the T, B plane. We know $\langle T(s), \vec{v} \rangle = \cos\theta$, so $\langle T(s), B(s) \rangle = 0 \Rightarrow \langle B(s), \vec{v} \rangle = -\sin\theta$.



$$-\sin\theta = \cos\left(\frac{\pi}{2}+\theta\right)$$

or

$$+\sin\theta = \cos\left(\frac{\pi}{2}-\theta\right)$$

Thus

$$-\kappa(s)\cos\theta \mp \tau(s)\sin\theta = 0$$

and

$$\frac{\tau(s)}{\kappa(s)} = \pm\frac{\cos\theta}{\sin\theta} = \pm\cot\theta, \text{ which is constant! } \blacksquare$$

($\Leftarrow$) Given that $\frac{\tau(s)}{\kappa(s)}$ is constant, let it equal $\cot\theta$, and set

$$\vec{v}(s) = \cos\theta\, T(s) + \sin\theta\, B(s)$$

We'll then compute

$$\vec{v}'(s) = (\kappa(s)\cos\theta - \tau(s)\sin\theta)\, N(s)$$

$= 0$ work it out, but comes from cross multiplying in

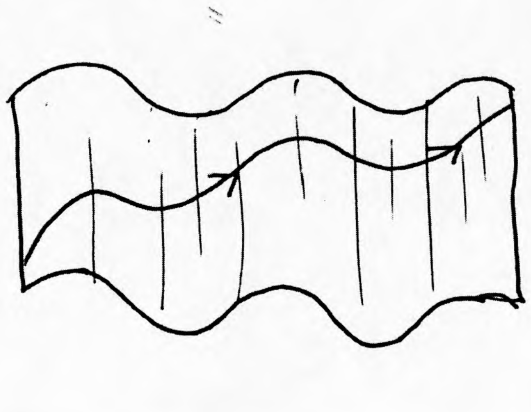
$$\frac{\tau(s)}{\kappa(s)} = \frac{\cos(\theta)}{\sin(\theta)}.$$

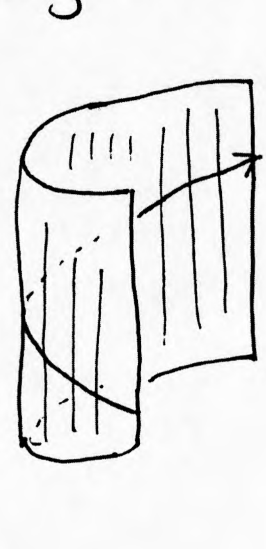
so $\vec{v}$ is a constant vector. But

$\langle \vec{v}, T(s) \rangle = \cos\theta$, which is constant. $\square$

Note that we ~~used~~ got $\kappa \neq 0$ in the $(\Leftarrow)$ part of the proof "for free" because the ratio $\tau(s)/\kappa(s)$ existed.

A generalized helix actually lies on a flat surface formed by extending the $\vec{v}$ direction.





So what if we fix

$$\kappa(s) = c \quad \underline{\text{or}} \quad \tau(s) = c$$

and let the other function vary as you like? Do we learn anything about the curve? Very little!

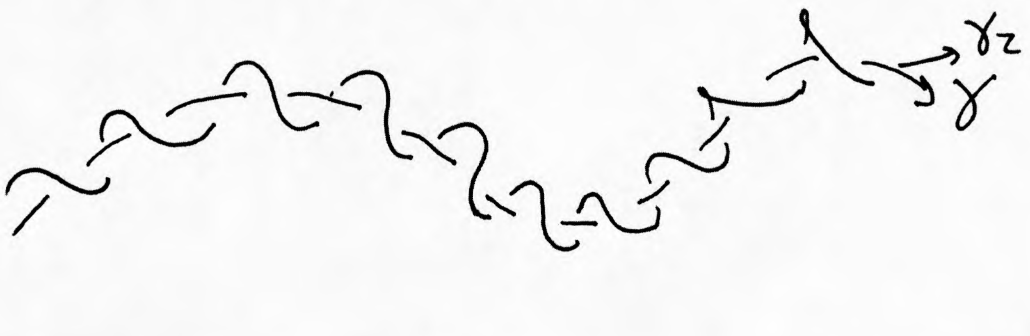
Theorem [Ghomi, 2006]

If γ is a curve of maximum curvature K and $K_2 \geq K$, then for any $\epsilon > 0$ $\exists$ a curve γ_2 of constant curvature K_2 so that

$$|\gamma(s) - \gamma_2(s)| < \epsilon \quad \underline{\text{and}} \quad |\gamma'(s) - \gamma_2(s)| < \epsilon$$

for all s.

A similar statement holds for curves of constant torsion.



We finish with a (surprising) open question:

Find conditions on $\kappa(t)$ and $\tau(t)$ which imply that the curve with this curvature and torsion is closed.