Polygons and Knots

Jason Cantarella

University of Georgia

UnKnot Conference, Denison University
August 2, 2016

Big Idea

Suppose we replace curves by n-edge polygons. This gives us a finite-dimensional version of knot theory where we can more easily understand the topology and the geometry of curve space.

Theorem (Kuiper, Randell 1987)

Every polygonal knot has at least 6 edges. (And there are 6 edge polygonal knots.)

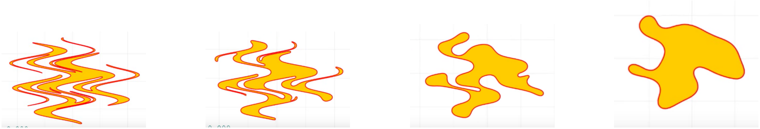
Theorem (Jin)

If $2 \leq p < q < 2p$, every polygonal (p, q) torus knot has at least $2q$ edges (and there at $2q$ edge polygonal (p, q) torus knots).

Two-dimensional Knot Theory

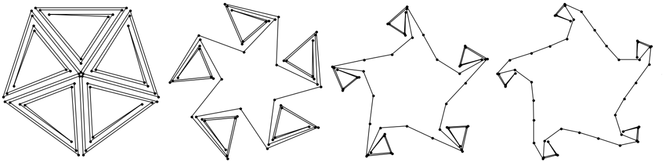
Theorem (1930's?)

There are no smooth planar knots.



Theorem (Connelly, Demaine, Rote 2004)

There are no polygonal planar knots.



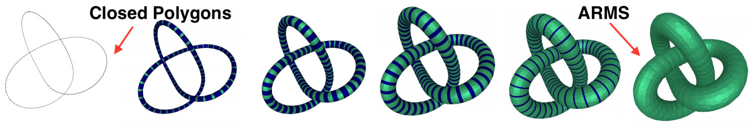
The Structure of Planar Polygons of Fixed Edgelength

The space of planar polygons with fixed edgelengths $w_1, \ldots, w_n$ can be described in two ways:

- the set of all configurations of n points $\vec{p}_1, \ldots, \vec{p}_n$ in $\mathbb{R}^2$ such that
$$|\vec{p}_{i+1} - \vec{p}_i| = w_i$$
- the set of all configurations of n angles θ_i on the circle so that the weighted average
$$\sum w_i(\cos\theta_i, \sin\theta_i) = \vec{0}$$

For this talk, we'll focus on $w_i = 1$, but the theory isn't really different otherwise.

We now want to do an Eric Rawdon type thing– how do we close polygons? If we could continuously deform every open polygon to a closed polygon, we'd have proved that open and closed polygons have the same topology.



Theorem (Farber, Walker, etc)

Well, they don't.

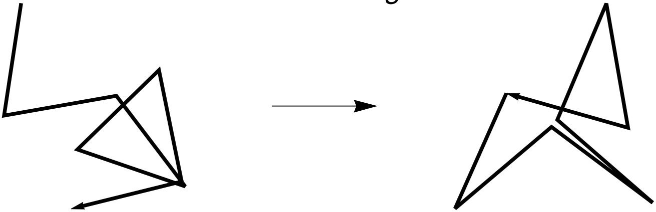
So let's go for something a little weaker:

Definition

The set of non-closed polygons where no more than half of θ_i are equal is called the *stable* polygonal arms.

Theorem (Kapovich-Millson)

There is a continuous map which associates each stable equilateral polygonal arm uniquely with a closed equilateral polygon with the same number of edges.



Definition

The maps (hyperbolic isometries!)

$$f(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$$

form a group of conformal (angle-preserving), orientation-preserving transformations of the unit disk to itself called PSL$(2,\mathbb{R})$ which preserves the unit circle.

Theorem (Kapovich-Millson, Sjamaar)

- *For every stable equilateral arm A, there is a unique $\mu(A)$ in the unit disk so that $c(z) = \frac{z-\mu}{1-\bar{\mu}z}$ maps z_i to a closed polygon $c(z_i)$.*
- *The closed polygon $c(z_i)$ is uniquely associated to the family of arms related to z_i by elements of $\mathrm{PSL}(2, \mathbb{R})$.*
- *The $c(z_i)$ is the polygon you'd get from flowing along the gradient of end-to-end distance squared.*
- *Closed equilateral polygons are the GIT quotient of $SO(2)^n$ by $\mathrm{PSL}(2, \mathbb{R})$.*

Question

Is $\mu(A)$ the center of mass of the z_i? If so, $c(z)$ would certainly take the center of mass of the z_i to the origin.

Proposition

Given a collection of z_i on the unit circle with center of mass $w = \frac{1}{n}\sum z_i$, the Milnor-Abikoff-Ye iteration

$$z_i \mapsto \frac{z_i - w}{1 - \bar{w} z_i}$$

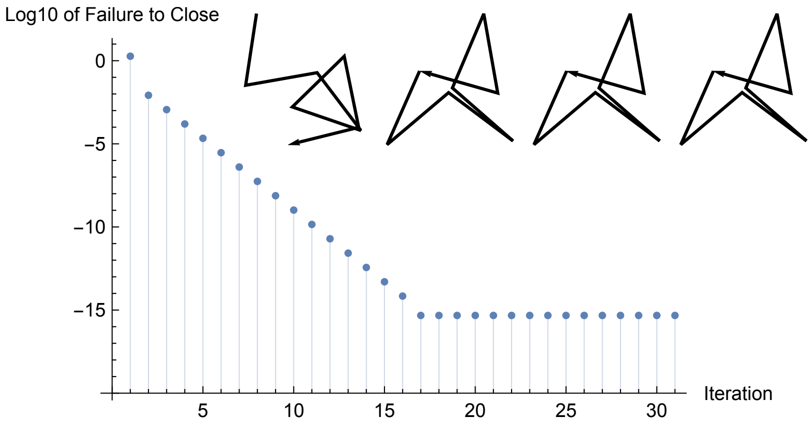
yields a point cloud $f(z_i)$ with center of mass $w' = \frac{1}{n}\sum f(z_i)$ satisfying

$$|w'| < \alpha |w|$$

where $\alpha < 1$ depends only on n.

Proposition (with Shonkwiler)

MAY iteration converges linearly to the closed polygon $c(z)$.



Definition

The elliptic distance energy of a polygon is the sum

$$E(P) = \sum_{\text{vertex } p_i, \text{edge } e_{jk}} \frac{1}{\underbrace{(|p_i - p_j| + |p_i - p_k| - |p_j - p_k|)^2}_{\text{the triangle inequality } \implies \geq 0}}$$

Proposition (with Demaine, O'Brien)

If a vertex of the polygon is on an edge, $E(P)$ is infinite. The only critical points of $E(P)$ are convex polygons.

Algorithm for planar polygonal unknotting

Algorithm

Given a configuration of a planar equilateral polygon P:

1. *Find the negative gradient vector of E at P, a set of velocities v_i for angles θ_i.*
2. *Construct the family of (open) polygons $p_i(\lambda) = \theta_i + \lambda v_i$.*
3. *Define $e(\lambda)$ to be the elliptic distance energy of the closed polygon $c(p_i(\lambda))$. Use MAY iteration to compute $e(\lambda)$ fast.*
4. *Search for a local min λ_0 of $e(\lambda)$; $e(\lambda_0) < e(0) = E(P)$. Update P to $P' = e(\lambda_0)$.*

Repeat until P is sufficiently close to a convex polygon.

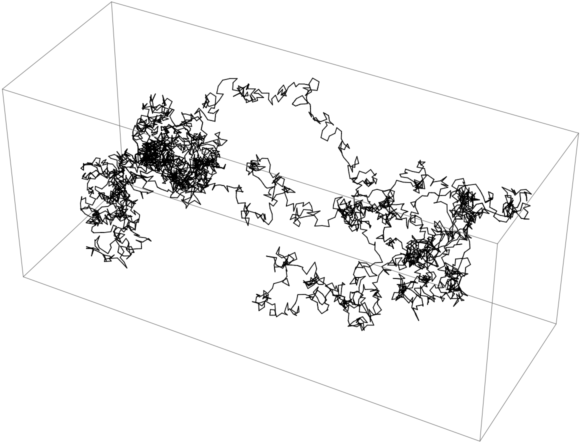
Theorem-in-progress (with Shonkwiler)

Time bounds.

Demonstration

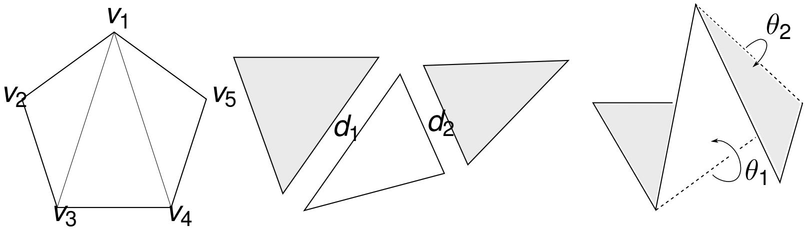
Definition

An equilateral space polygon is given by a collection of edge directions e_i on S^2 which sum to zero.



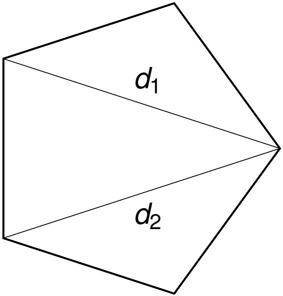
Definition

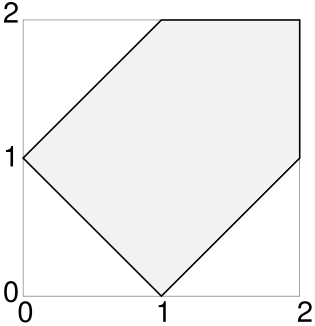
If we triangulate a regular n-gon, we see the polygon is the boundary of a folded structure made of $n-3$ flat triangles joined at $n-3$ hinges. The lengths d_i and angles θ_i are a system of coordinates on polygon space called action-angle coordinates.



Definition

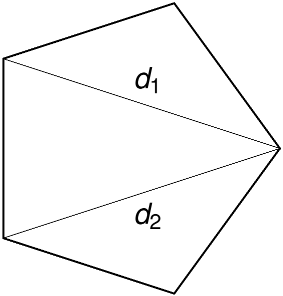
The hinge lengths are sides of triangles, so they obey some triangle inequalities which define a convex polytope in $\mathbb{R}^{n-3}$ called the *triangulation polytope* $\mathcal{P}_n(\vec{r})$.

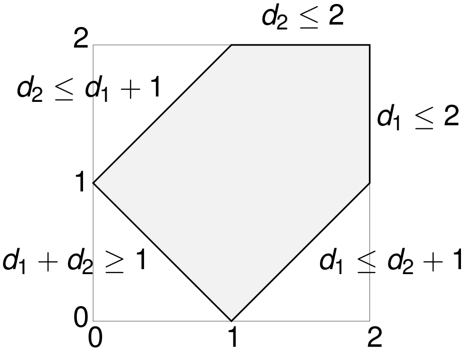




Definition

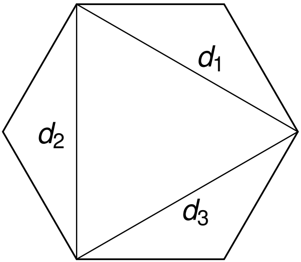
The hinge lengths are sides of triangles, so they obey some triangle inequalities which define a convex polytope in $\mathbb{R}^{n-3}$ called the *triangulation polytope* $\mathcal{P}_n(\vec{r})$.

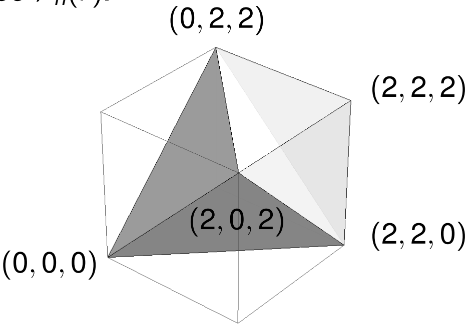




Definition

The hinge lengths are sides of triangles, so they obey some triangle inequalities which define a convex polytope in $\mathbb{R}^{n-3}$ called the *triangulation polytope* $\mathcal{P}_n(\vec{r})$.





Why This Function?

Theorem (Archimedes, Duistermaat–Heckman)

The action-angle coordinates θ, z on the sphere given by

$$(\theta, z) \mapsto (\sqrt{1 - z^2}\cos\theta, \sqrt{1 - z^2}\sin\theta, z)$$

are an area-preserving map from the cylinder to the sphere.

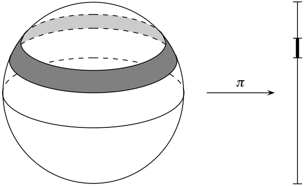


Illustration by Kuperberg.

Theorem (with Shonkwiler)

Action-angle coordinates are a volume-preserving map from torus $\times$ *polytope (with Euclidean volume!) to the space of closed polygons.*

Corollary

Probabilities computed over torus $\times$ *polytope equal probabilities computed over polygon space.*

Proposition (with Shonkwiler)

The expected length of a chord skipping k edges in an n-edge equilateral polygon is the $(k-1)$*st coordinate of the center of mass of the moment polytope for* $\mathrm{Pol}(n; \vec{1})$*.*

Proposition (with Shonkwiler)

The expected length of a chord skipping k edges in an n-edge equilateral polygon is the $(k-1)$st coordinate of the center of mass of the moment polytope for $\mathrm{Pol}(n;\vec{1})$.

n	$k=2$	3	4	5	6	7	8
4	1						
5	$\frac{17}{15}$	$\frac{17}{15}$					
6	$\frac{14}{12}$	$\frac{15}{12}$	$\frac{14}{12}$				
7	$\frac{461}{385}$	$\frac{506}{385}$	$\frac{506}{385}$	$\frac{461}{385}$			
8	$\frac{1{,}168}{960}$	$\frac{1{,}307}{960}$	$\frac{1{,}344}{960}$	$\frac{1{,}307}{960}$	$\frac{1{,}168}{960}$		
9	$\frac{112{,}121}{91{,}035}$	$\frac{127{,}059}{91{,}035}$	$\frac{133{,}337}{91{,}035}$	$\frac{133{,}337}{91{,}035}$	$\frac{127{,}059}{91{,}035}$	$\frac{112{,}121}{91{,}035}$	

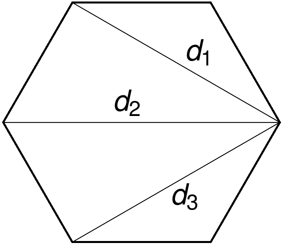
Proposition (with Shonkwiler)

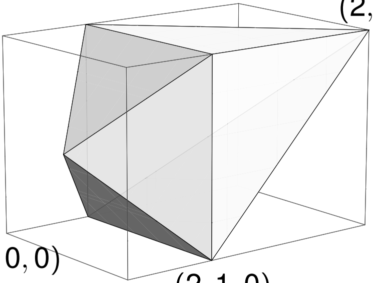
The expected length of a chord skipping k edges in an n-edge equilateral polygon is the $(k-1)$st coordinate of the center of mass of the moment polytope for $\mathrm{Pol}(n;\vec{1})$.

$$E(\mathrm{chord}(37,112)) = \frac{\begin{array}{l}2586147629602481872372707134354784581828166239735638\\0021498840205773666873699649081859732772942937515338\\2121765570397854911152980222231191532164599823845519\\58079667505955874840298583338222480954393259655695\end{array}}{\begin{array}{l}5610189772922960964198156790682037660099932612686267\\0741808227567749566915324470667755069070793713602742\\4519117786555575048213829170264569628637315477158\\307368641045097103310496820323457318243992395055104\end{array}}$$

$$\approx 4.60973$$

The Fan Triangulation Polytope





The polytope $\mathcal{P}_n = \mathcal{P}_n(\vec{1})$ corresponding to the "fan triangulation" is defined by the triangle inequalities:

$$0 \leq d_1 \leq 2 \qquad \begin{array}{c} 1 \leq d_i + d_{i+1} \\ |d_i - d_{i+1}| \leq 1 \end{array} \qquad 0 \leq d_{n-3} \leq 2$$

If we introduce a fake chordlength $d_0 = 1 = d_{n-2}$, and make the linear transformation

$$s_i = d_i - d_{i-1}, \text{ for } 1 \leq i \leq n-2$$

then $\sum s_i = d_{n-2} - d_0 = 0$, so s_{n-2} is determined by $s_1, \ldots, s_{n-3}$

A Change of Coordinates

If we introduce a fake chordlength $d_0 = 1 = d_{n-2}$, and make the linear transformation

$$s_i = d_i - d_{i-1}, \text{ for } 1 \le i \le n-2$$

then $\sum s_i = d_{n-2} - d_0 = 0$, so s_{n-2} is determined by $s_1, \ldots, s_{n-3}$ and the inequalities

$$0 \le d_1 \le 2 \qquad \begin{array}{c} 1 \le d_i + d_{i+1} \\ |d_i - d_{i+1}| \le 1 \end{array} \qquad 0 \le d_{n-3} \le 2$$

become

$$\underbrace{-1 \le s_i \le 1}_{\text{easy conditions}}, \qquad \underbrace{-1 \le \sum_{i=1}^{n-3} s_i \le 1 \quad \sum_{j=1}^{i} s_j + \sum_{j=1}^{i+1} s_j \ge -1}_{\text{harder conditions}}$$

A fantastically stupid idea (that works great!)

Just pick $s_1, \ldots, s_{n-3}$ independently in $[-1, 1]$, so we know they obey the easy conditions and throw out any picks which don't obey the hard conditions

$$-1 \leq \sum_{i=1}^{n-3} s_i \leq 1, \quad \sum_{j=1}^{i} s_j + \sum_{j=1}^{i+1} s_j \geq -1.$$

Theorem (with Duplantier, Shonkwiler, Uehara)

For large n the expected number of tries before a success is asymptotic to

$$\frac{\sqrt{\pi}}{6\sqrt{6}} n^{3/2}.$$

We can now state the final sampling algorithm, which generates perfect, independent samples of polygon space:

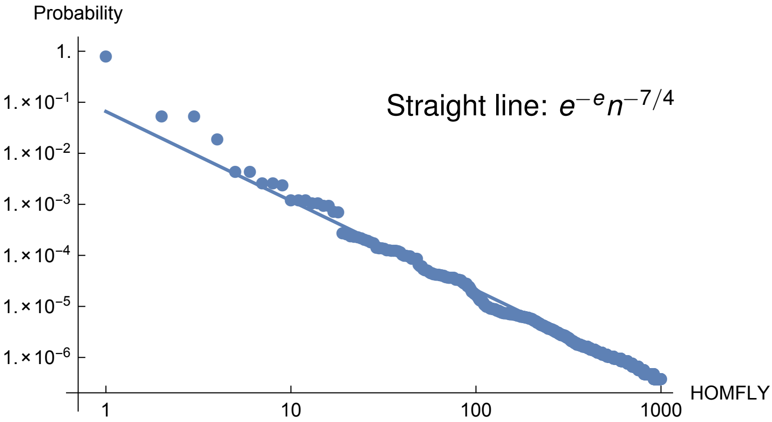
Action-Angle Method (with Duplantier, Shonkwiler, Uehara 2015)

1. *Generate* $(s_1, \ldots, s_{n-3})$ *uniformly on* $[-1, 1]^{n-3}$ *until they obey hard inequalities, build corresponding* $d_1, \ldots, d_n$*.*
2. *Generate dihedral angles* $\theta_1, \ldots, \theta_{n-3}$*.*
3. *Build polygon from action-angle coordinates.*

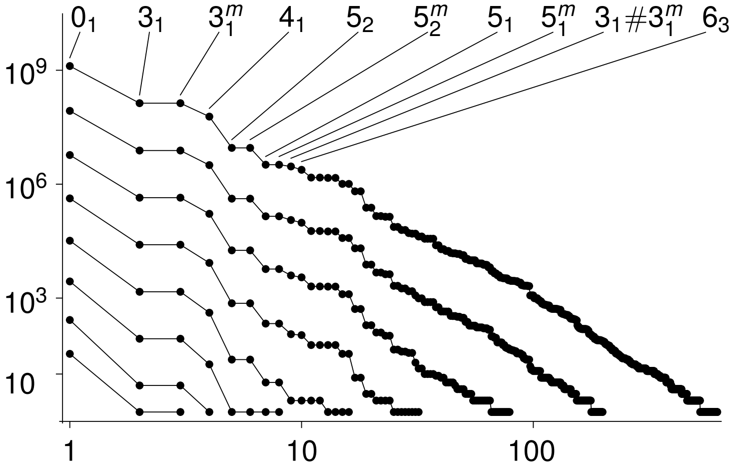
We've proved above that this takes $O(n^{5/2})$ time per sample.

Rank stats for 60-gons

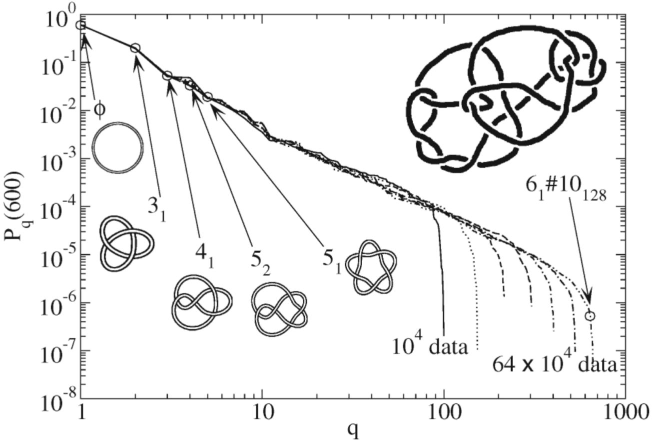
How common are various kinds of knots among all polygons?



Corresponding graph in the random diagram model:



Rank stats for 600 edge collapsed SAP model



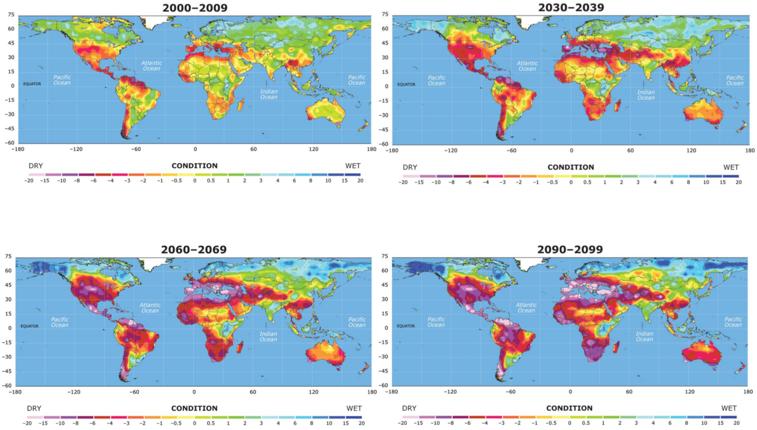
Baiesi, Orlandini, Stasiak 2007

Current Questions

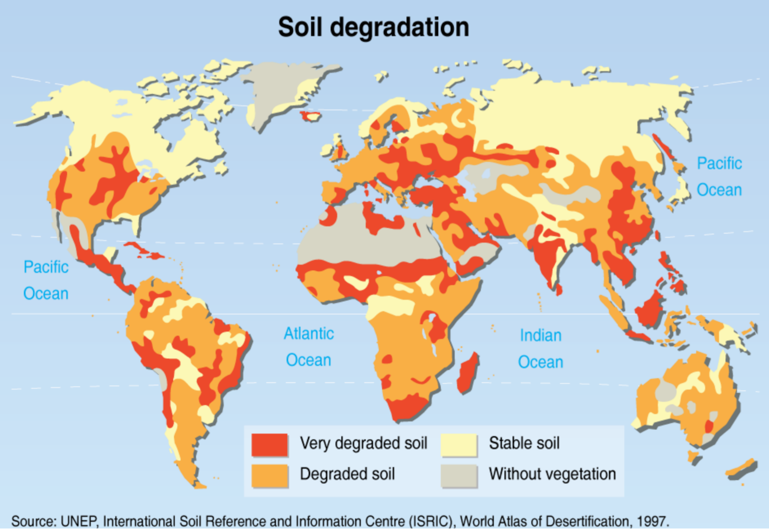
- What's up with the linear rank statistics?
- We can find paths between equilateral space polygons. How about geodesics? (cf. my student Tom Needham's work on curves– solves the problem for nonequilateral polygons)
- Can we interpolate between polygons? Define the "average shape" of a collection of curves?
- Can we generalize all this to a statistics of theta curves? Handcuff graphs? Trees?
- Trees?
- Trees?

Why am I so fixated on trees?

Global climate change will intensify drought in this century



A public service announcement



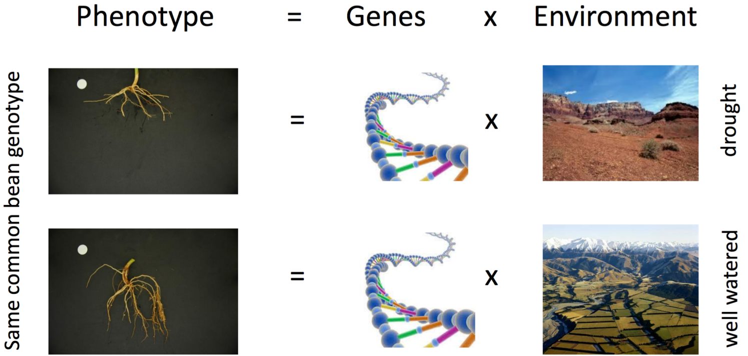
Projection

***Per-acre** crop yields must double by 2050, or people starve.*
On worse farmland.
With less water, less fertilizer, and less agricultural chemicals.

- *Tilman, D., et al.,* ***Global food demand and the sustainable intensification of agriculture****. Proceedings of the National Academy of Sciences, 2011. 108(50): p. 20260-20264*
- *Ray DK, Mueller ND, West PC, Foley JA.* ***Yield Trends Are Insufficient to Double Global Crop Production by 2050.*** *Hart JP, ed. PLoS ONE. 2013;8(6):e66428. doi:10.1371/journal.pone.0066428.*

Problem: root system bioengineering

How to improve roots?



Assumption: If an observed root trait variation is linked to genes, than the trait is possible to breed

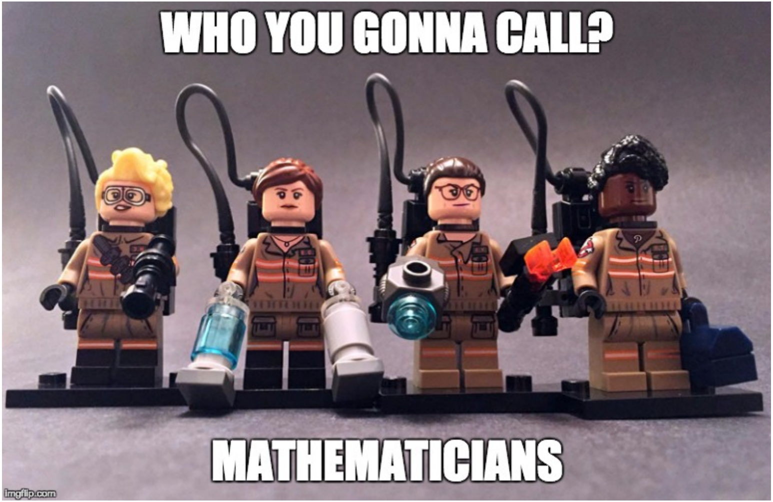
Math problem: DiRT, collaboration with Buksch lab

Unknown which genes affect roots. Large databases of digital photography of root systems show significant variations between root systems, even for cloned plants.

Problem

Find a mathematically defensible theoretical framework for measuring the significance of geometric differences between populations of tree shapes.

The molecular biologists are just guessing until we solve it, the computer scientists already did their stuff . . . and it *matters*.



Thank you for listening!

References

- *The Symplectic Geometry of Closed Equilateral Random Walks in 3-Space*
  Jason Cantarella and Clayton Shonkwiler
  Annals of Applied Probability 26 (2016), no. 1, p. 549-596

- *A Fast Direct Sampling Algorithm for Equilateral Closed Polygons*
  Jason Cantarella, Bertrand Duplantier, Clayton Shonkwiler, and Erica Uehara
  Journal of Physics A 49 (2016), no. 27, p. 275205.