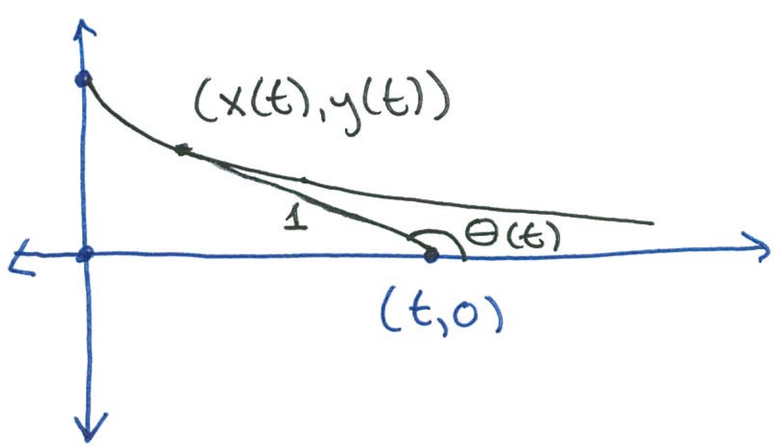
The tractrix.

A mass is located at (0,1) and pulled by a linkage of fixed length 1 moving along the x-axis at speed 1.

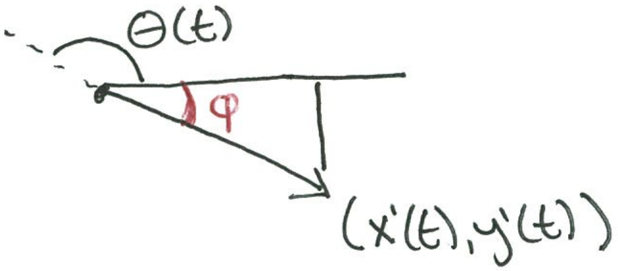


We know

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

because of the length-1 constraint.

Less obviously, the linkage is tangent to the curve, so we know that we have a triangle



Since

$$\tan \varphi = \frac{-y'(t)}{x'(t)}$$ ↙ recall $y'(t)$ is negative!

and $\varphi = \pi - \theta$, the supplementary angle formula for tan tells us that

$$\tan \theta = \frac{y'(t)}{x'(t)} = \frac{\cos\theta \, \theta'(t)}{1 - \sin\theta \, \theta'(t)}$$

We can solve this formula for $\theta'(t)$.

$$\tan\theta\,(1 - \sin\theta\,\theta') = \cos\theta\,\theta'$$

$$\tan\theta - \tan\theta\sin\theta\,\theta' = \cos\theta\,\theta'$$

$$\tan\theta = \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)\theta'$$

→ multiplying through by cos.

$$\sin\theta = \theta'$$

We can solve this by separation of variables: $\sin\theta = \frac{d\theta}{dt}$, so

$$\int \frac{1}{\sin\theta}\,d\theta = \int 1\,dt$$

~~and~~ or

$$\int \csc\theta\,d\theta = -\ln(\csc\theta + \cot\theta) + C$$
$$= t$$

for some constant C.

at $t=0$, we have $\theta = \pi/2$, so

$$\csc \pi/2 = 1, \quad \cot \pi/2 = \frac{0}{1} = 0$$

and $-\ln(\csc \pi/2 + \cot \pi/2) = -\ln 1 = 0$.

This means $c=0$. So

$$t = -\ln\left(\csc\theta + \cot\theta\right).$$

We want to solve this for θ.

Now

$$\csc\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta}$$

We know

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\sin\theta = 2\cos\theta/2\,\sin\theta/2$$

so

$$\frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2\theta/2}{2\cos\theta/2\,\sin\theta/2}$$

$$= \cot\theta/2$$

(who said trig was useless!?) and

$$t = +\ln\tan\theta/2$$

we switched from cot to tan, killing the minus sign.

so the tractrix is parametrized by θ if we substitute this back into

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

to get

$$x(\theta) = \cos\theta + \ln \tan \theta/2$$
$$y(\theta) = \sin\theta$$

Looking at start, end we see

$$\pi/2 \leq \theta < \pi$$

What about a t parametrization? Well, exp-ing $t = \ln \tan \theta/2$, we get

$$e^t = \tan \theta/2$$

We now have to solve for $\sin\theta$ and $\cos\theta$ in terms of $\tan \theta/2$.

This is a trig exercise very similar to what we've already done.

Recall

$$\sin\theta = 2\sin\theta/2\,\cos\theta/2$$

$$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$$

and that we can use the sin and cos in terms of tan formulas to write

$$\sin\theta/2 = \frac{\tan\theta/2}{\sqrt{1+\tan^2\theta/2}}$$

$$\cos\theta/2 = \frac{1}{\sqrt{1+\tan^2\theta/2}}$$

so we have

$$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$$

Now plugging in $e^t = \tan\theta/2$,

we get

$$\sin\theta = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^{-t}+e^t} = \operatorname{sech} t$$

and

$$\cos\theta = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t}-e^t}{e^{-t}+e^t} = -\tanh t$$

so we can also parametrize the tractrix by

$$(t - \tanh t, \operatorname{sech} t), \quad t \geq 0.$$