The Bishop Frame.

We now define another ~~version of~~ curve framing.

Suppose we start with some (arbitrary) smooth framing F of γ, and consider the vector field

$$V(s) = \cos\theta(s)\, F_2(s) + \sin\theta\, F_3(s)$$

Then

$$\begin{aligned} V'(s) = & -\sin\theta(s) \cdot \theta'(s)\, F_2(s) \\ & + \cos\theta(s)\left(-\alpha_{12}(s) F_1(s) + \alpha_{23}(s) F_3(s)\right) \\ & + \cos\theta(s) \cdot \theta'(s)\, F_3(s) \\ & + \sin\theta\left(-\alpha_{13}(s) F_1(s) - \alpha_{23} F_2(s)\right) \end{aligned}$$

Gathering terms,

$$\begin{aligned} V'(s) = & \left(-\alpha_{12}(s)\cos\theta(s) - \alpha_{13}(s)\sin\theta(s)\right) F_1(s) \\ & \left(-\alpha_{23}(s)\sin\theta(s) - \sin\theta(s)\,\theta'(s)\right) F_2(s) \\ & \left(+\alpha_{23}(s)\cos\theta(s) + \cos\theta(s)\,\theta'(s)\right) F_3(s) \end{aligned}$$

Observe that the F_2, F_3 coefficients are

$$-\sin\theta(s)\left(\alpha_{23}(s)+\theta'(s)\right)$$

and

$$\cos\theta(s)\left(\alpha_{23}(s)+\theta'(s)\right)$$

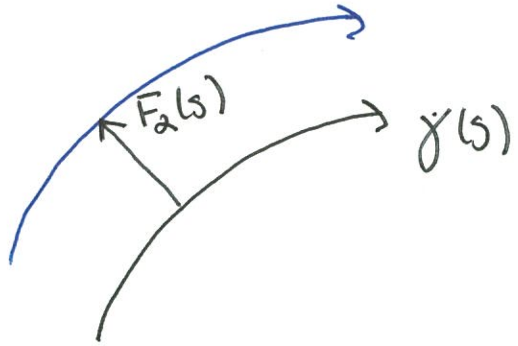
This means that if we set $\theta'(s) = -\alpha_{23}(s)$ and integrate, we can define a family of frames such that:

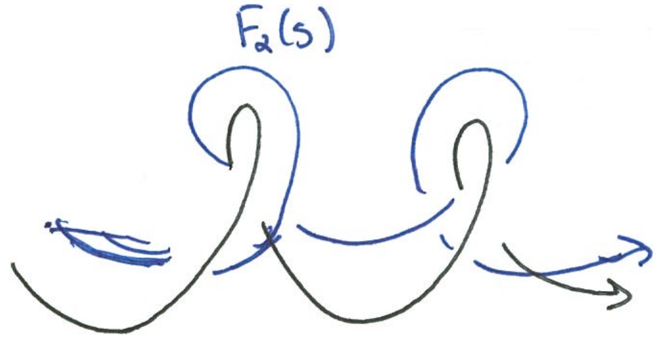
$F_2(s) = V_2(s)$ has $V'(s) = \lambda(s)\,T(s)$

$F_2(s)$ depends on the initial angle $\theta(0) = \theta_0$, but any two such frames with initial difference in angle $\Delta\theta_0$ maintains this angular difference for all s.

We call this construction the Bishop frame, or relatively parallel adapted frame (RPAF).

The picture is





Traditionally, we write the structure equations

$$
\begin{aligned}
T' &= \kappa_1 F_2 + \kappa_2 F_3 \\
F_2' &= -\kappa_1 F_2 \\
F_3' &= -\kappa_2 F_3
\end{aligned}
$$

and call $\alpha_{12}(s) = \kappa_1(s)$ and $\alpha_{13}(s) = \kappa_2(s)$.

These two functions are like κ, τ:

$$
\begin{aligned}
|T'| &= \kappa(s) \\
&= \sqrt{\kappa_1(s)^2 + \kappa_2^2(s)},
\end{aligned}
$$

so $\kappa(s)$ is like radius in the κ_1, κ_2 plane.

To compute torsion, observe

$$N(s) = \frac{T'(s)}{|T'(s)|} = \underbrace{\left(\frac{K_1}{K}\right)}\, F_2 + \underbrace{\left(\frac{K_2}{K}\right)} F_3$$

two numbers, squares sum to 1

$$= \cos\theta \, F_2 + \sin\theta \, F_3$$

Then

$$N'(s) \cdot B(s) = \tau(s)$$

$$= \langle \theta'(s) \overbrace{\left(-\sin\theta \, F_2 + \cos\theta \, F_3\right)}^{\text{this is } B}$$

$$+ \text{ something in } T \text{ direction}, B \rangle$$

$$= \theta'(s).$$

So

$$\tau(s) = \theta'(s), \quad \text{or} \quad \theta(s) = \int \tau(s)\, ds$$

Since $\frac{K_1}{\sqrt{K_1^2+K_2^2}} = \cos\theta$, $\frac{K_2}{\sqrt{K_1^2+K_2^2}} = \sin\theta$, we see

$\int \tau(s)\, ds$ is the polar θ in the K_1, K_2 plane.

We can now see to what extent κ, τ or κ_1, κ_2 determine the geometry of the curve γ!

We already know:

$\tau = 0 \iff \gamma$ is planar (and has nonvanishing κ)

In our new language, $\tau = 0 \iff (\kappa_1, \kappa_2)$ is on a line through the origin.

Proposition (Bishop, 1990's).
γ lies on a sphere $\iff (\kappa_1, \kappa_2)$ lies on a line not through the origin.

Proof. ~~Suppose~~
($\Rightarrow$) Wlog, we may assume the sphere is centered at the origin and $\langle \gamma(s), \gamma(s) \rangle = r^2$.
Differentiating, we see

$$\langle \gamma, \gamma' \rangle = 0.$$

Here's a neat trick. Choose any Bishop frame (T, F_2, F_3) on γ. For any s, this is a basis for $\mathbb{R}^3$, so we can write $\gamma(s)$ in this basis.

In principle,

$$\gamma(s) = \lambda_1(s)T(s) + \lambda_2(s)F_2(s) + \lambda_3(s)F_3(s)$$

but we know $\langle \gamma(s), T(s) \rangle = 0$, so $\lambda_1(s) \equiv 0$.

This leaves us with

$$\gamma = \lambda_2 F_2 + \lambda_3 F_3$$

so

$$\gamma' = \lambda_2' F_2 + \lambda_2 k_1 T + \lambda_3' F_3 + \lambda_3 k_2 T$$

or

$$\vec{0} = (\lambda_2 k_1 + \lambda_3 k_2 - 1)T + \lambda_2' F_2 + \lambda_3' F_3$$

But this means $\lambda_2' = 0$, $\lambda_3' = 0$ so the λ_i are constants.

And then we have

$$\lambda_2 K_1(s) + \lambda_3 K_2(s) = 1$$

which is exactly the equation of a line not through the origin! $\square$

($\Leftarrow$) Suppose $\lambda_2 K_1(s) + \lambda_3 K_2(s) = 1$ for some constants λ_2, λ_3. Choose a Bishop frame and Consider the vector

$$\gamma(s) - \left(\lambda_2 F_2(s) + \lambda_3 F_3(s)\right) = \alpha(s).$$

If we differentiate,

$$\begin{aligned}\alpha'(s) &= T(s) - \lambda_2 K_1(s) T(s) - \lambda_3 K_2(s) T(s) \\ &= \left(1 - \lambda_2 K_1(s) - \lambda_3 K_2(s)\right) T(s) \\ &= 0.\end{aligned}$$

So $\alpha(s)$ is a constant vector, $\vec{\alpha}$. We claim this is the center of the sphere.

To prove it,

$$\frac{d}{ds}\langle \gamma(s)-\alpha, \gamma(s)-\alpha\rangle =$$

$$= 2\langle T(s), \gamma(s)-\alpha\rangle$$

$$= 2\langle T(s), \lambda_2 F_2(s) + \lambda_3 F_3(s)\rangle$$

$$= 0$$

So γ is on a sphere centered at α. $\square$