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A New Proof of Laman's Theorem

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Abstract. We give a simple proof for the characterization of generically rigid bar frameworks in the plane.

1. Introduction

Crapo gave a characterization of the generic rigidity of bar frameworks in the plane in terms of tree decompositions (H.H. Crapo, On the generic rigidity of plane frameworks, preprint):

A graph G is realizable as a generically rigid bar framework in the plane if and only if it contains three edge disjoint trees T_1, T_2, T_3, such that every vertex of G is incident with exactly two of the trees and distinct subtrees (*with at least one edge*) *of the trees T_i do not have the same vertex set.*

Crapo proved this by using Laman's Theorem [1] which characterizes generic rigidity of bar frameworks in the plane. In this article we give a direct proof of this result. The important thing is that this direct proof may be adapted to give a short proof of the characterization of the generic rigidity of $(n-1, 2)$-frameworks in n space [2, 3]. This structure consists of a set of $(n-2)$-dimensional panels where certain pairs are linked by one or more rods (line segments) using ball joints.

2. Basic Definitions

A *graph* $G = (V, E)$ consists of a finite set of *vertices* V and a set of *edges* E whose elements are 2-element subsets of V. Vertices are usually denoted by lower case Roman letters. An edge $\{a, b\}$ is usually denoted by ab and sometimes by just a single lower case Greek letter. For any $A \subseteq V$ we shall use E_A to denote the set of edges which are incident to only vertices in A. For any disjoint $A, B \subseteq V$, $E_{A,B}$ shall denote the set of edges incident with a vertex in A and a vertex in B. We use $\langle V' \rangle$

to denote the subgraph induced by V', i.e., $\langle V' \rangle$ is the subgraph $(V', E_{V'})$. Let $G' = (V', E')$ and $G^* = (V^*, E^*)$ be subgraphs of G. The *intersection* $G' \cap G^*$ of G' with G^* is the subgraph $(V' \cap V^*, E' \cap E^*)$. Their *union* $G' \cup G^*$ is the subgraph $(V' \cup V^*, E' \cup E^*)$. For a set A of edges of vertices, we shall use the corresponding lower case letter a to denote its cardinality. Also the graph G is always $G(V, E)$.

An nTk *graph* (G, T_i) is a graph G whose edge set is expressed as a disjoint union of n trees T_i such that every vertex of G is in precisely k of these trees. An nTk graph is *proper* if for every $V' \subseteq V$ such that $v' \geq 2$ at most $k - 1$ of the intersections $\langle V' \rangle \cap T_i$ are trees with vertex set V'. A graph G has an nTk *decomposition* if there exist n trees T_i such that (G, T_i) is an nTk graph.

3. Bar Frameworks in 2-space

A *generalized bar framework* $(G, \mathbf{p}, \mathbf{q})$ in the plane is a graph G together with functions $\mathbf{p}\colon V(G) \to \mathbb{R}^2$ and $\mathbf{q}\colon E(G) \to \mathbb{R}^2$ such that

$$\mathbf{p}_a - \mathbf{p}_b = \pm k_{ab}\mathbf{q}_{ab}$$

where k_{ab} is a constant which is possibly zero. A *bar framework* is then a generalized bar framework satisfying $\mathbf{p}_a \neq \mathbf{p}_b$ if ab is an edge.

The *rigidity matrix* $\mathbf{R}(G, \mathbf{p}, \mathbf{q})$ of a generalized bar framework $(G, \mathbf{p}, \mathbf{q})$ is an $e \times v$ matrix with entries in $\mathbb{R}^2$ whose rows are indexed by $E(G)$ and whose columns are indexed by $V(G)$. We assume the vertices and edges are in an arbitrary, but otherwise fixed, linear order. The rows and columns are arranged according to this order. If ab is an edge with $a < b$ in the given linear order, then a is the *first vertex* and b the *last vertex* of ab. The (α, a) entry is given by

$$\mathbf{R}_{\alpha,a} = \begin{cases} \mathbf{q}_\alpha & \text{if } a \text{ is the first vertex of } \alpha \\ -\mathbf{q}_\alpha & \text{if } a \text{ is the last vertex of } \alpha \\ \mathbf{0} & \text{otherwise} \end{cases}$$

$(G, \mathbf{p}, \mathbf{q})$ is *independent* if the rows of its rigidity matrix are independent. An *infinitesimal motion* of $G(\mathbf{p}, \mathbf{q})$ is a function $\mathbf{m}\colon \mathbf{V} \to \mathbb{R}^2$ satisfying $(\mathbf{m}_a - \mathbf{m}_b)\cdot \mathbf{q}_{ab} = 0$ for every edge ab. It follows that the space of motion can be interpreted as the orthogonal complement of the row space of $\mathbf{R}$. A motion is *trivial* if it can be extended to the Euclidean motion of the entire plane. A bar framework is *infinitesimally rigid* if all its motions are trivial. The graph G is generically rigid as a bar framework in the plane if there exist $\mathbf{p}$, $\mathbf{q}$, such that $(G, \mathbf{p}, \mathbf{q})$ is infinitesimally rigid. Since the dimension of the space of trivial motions is 3, we have the following two results.

Theorem 3.1. *A bar framework in the plane* $(G, \mathbf{p}, \mathbf{q})$ *(with at least* 2 *vertices) is infinitesimally rigid if and only if its rigidity matrix of rank* $2v - 3$.

Theorem 3.2. *Suppose G is generically independent as a bar framework in the plane, then for every subgraph $H = (V', E')$, we have $e' \leq 2v' - 3$.*

The following is the theorem of Laman [1].

Theorem 3.3. *A graph $G = (V, E)$ is generically rigid and independent as a bar framework in the plane if and only if for every subgraph $H = (V', E')$ with $v' \geq 2$, we have*

$$e' \leq 2v' - 3$$

and equality holds when $G = H$.

The condition in Laman's Theorem is known variously as Laman's condition, 2-count condition, etc. It is not hard to prove directly that a graph G has a proper 3T2 decomposition if and only if it satisfies Laman's condition, thereby establishing the following theorem (Theorem 3.4). This is basically the approach of Crapo. We give a direct proof.

Theorem 3.4. *A graph G is generically rigid and independent as a bar framework in the plane if and only if it has a proper 3T2 decomposition (G, T_i).*

Proof. Suppose there exist $\mathbf{p}$ and $\mathbf{q}$ such that $(G, \mathbf{p}, \mathbf{q})$ is infinitesimally rigid and independent. By Theorem 3.1 we have $e = 2v - 3$. This its rigidity matrix $\mathbf{R}$ has a $(2v - 3) \times (2v - 3)$ submatrix $\mathbf{R}'$ which is non-singular. We regard $\mathbf{R}$ as an $e \times 2v$ matrix with real entries. We can assume that $\mathbf{R}'$ does not include the 2 columns associated with the last vertex of G because these two columns are clearly dependent on the other columns. Thus we can assume that $\mathbf{R}'$ excludes the last three columns of $\mathbf{R}$. Since $\mathbf{R}'$ is non-singular, we can find two submatrices $\mathbf{R}_1, \mathbf{R}_2$ of $\mathbf{R}'$ whose rows are indexed by E_1 and E_2 with $e_1 = v - 1$ and $e_2 = v - 2$, respectively. The columns of $\mathbf{R}_1$ are the odd columns while those of $\mathbf{R}_2$ are the even columns of $\mathbf{R}'$. Moreover, $\det \mathbf{R}_1 \det \mathbf{R}_2 \neq 0$. If we append the last but one column of $\mathbf{R}$ to $\mathbf{R}_1$, and with proper scaling, we get the usual incidence matrix of the subgraph $\langle E_1 \rangle$. Since this matrix is of full rank, and $\langle E_1 \rangle$ has $v - 1$ edges and v vertices, it is a tree. Similarly, $\langle E_2 \rangle$ is the union of two edge disjoint trees. These trees give a 3T2 decomposition of the given graph. If this decomposition is not proper, then we can find subtrees T_1 of $\langle E_i \rangle$ and T_2 of $\langle E_j \rangle$, $i \neq j$, with the same set of vertices. Then $T_1 \cup T_2$ violates Theorem 3.2. So we have a proper 3T2 decomposition of the given graph.

For sufficiency, we suppose that (G, T_i), $i = 1, 2, 3$, is a proper 3T2 decomposition of G. Since G has $2v - 3$ edges it suffices to prove that there exist $\mathbf{p}$ and $\mathbf{q}$ such that the rows of $\mathbf{R}(G, \mathbf{p}, \mathbf{q})$ are linearly independent.

Let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $\mathbf{e}_3 = (0, 0)$, and $\mathbf{V}_i$ be the set of vertices that are not in T_i. Define $\mathbf{p}\colon V \to \mathbb{R}^2$ and $\mathbf{q}\colon E \to \mathbb{R}^2$ by

$$\mathbf{q}_\alpha = \begin{cases} \mathbf{e}_2 & \text{if } \alpha \in E(T_1) \\ \mathbf{e}_1 & \text{if } \alpha \in E(T_2) \\ \mathbf{e}_1 - \mathbf{e}_2 & \text{if } \alpha \in E(T_3) \end{cases}$$

$$\mathbf{p}_a = \mathbf{e}_i \qquad \text{if } a \in V_i$$

Then $(G, \mathbf{p}, \mathbf{q})$ is independent. However, for each $i = 1, 2, 3$, the vertices in V_i are in the same location. Since one of the V_i's may be empty, we may have only two

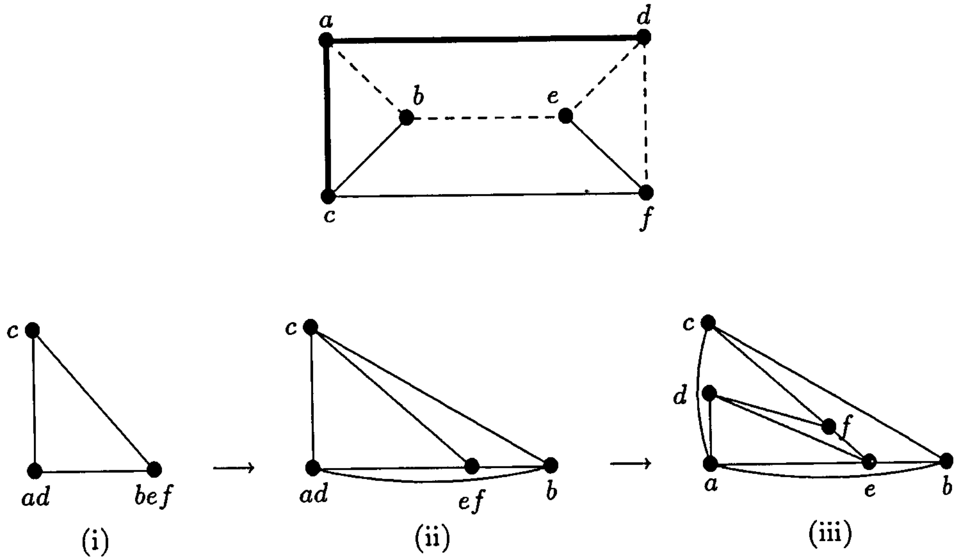
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Fig. 3.1

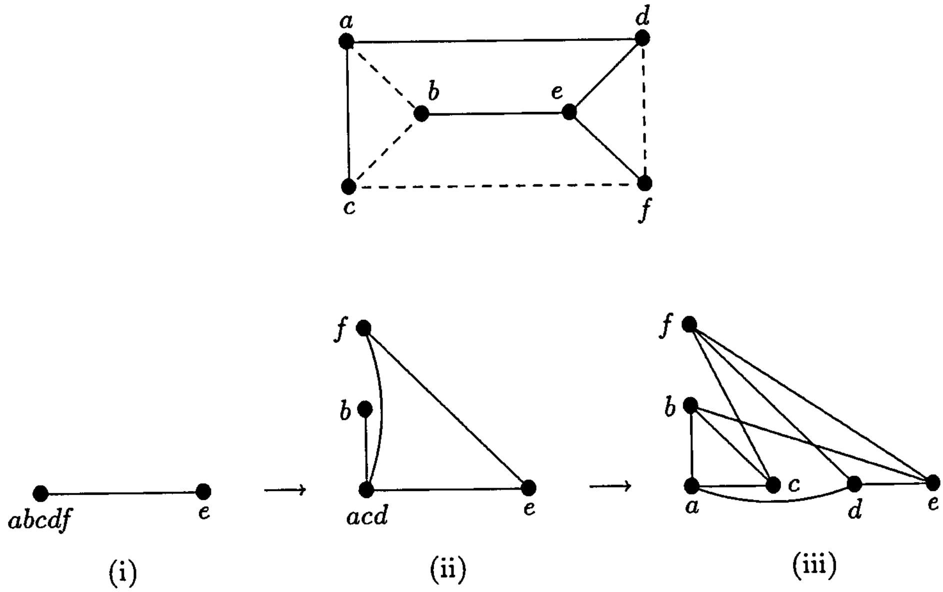
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Fig. 3.2

distinct locations for the vertices. Thus this is a generalized bar framework. We need to modify this realization so that the locations of the vertices are distinct.

Suppose $v_1 \geq 2$. Then one of $\langle V_1 \rangle \cap T_i$, $i = 2, 3$, say $i = 3$, is not connected because (G, T_i) is a proper $3T2$ decomposition. Let A be the set of vertices in one of the components of $\langle V_i \rangle \cap T_3$. Define $\mathbf{p}'$: $V(G) \to \mathbb{R}^2$ and $\mathbf{q}'$: $E(G) \to \mathbb{R}^2$ by

$$\mathbf{p}'_a = \begin{cases} (1 + t, 0) & \text{if } a \in A \\ \mathbf{p}_a & \text{otherwise} \end{cases}$$

$$\mathbf{q}'_\alpha = \begin{cases} (1 + t, -1) = \mathbf{q}_\alpha + (t, 0) & \text{if } \alpha \in E_{A,V_2} \\ \mathbf{q}_\alpha & \text{otherwise} \end{cases}$$

Then $(G, \mathbf{p}'(t), \mathbf{q}'(t)) = (G, \mathbf{p}, \mathbf{q})$ when $t = 0$. We now treat t as an indeterminate. The condition for the rows of $\mathbf{R}(G, \mathbf{p}', \mathbf{q}')$ to be linearly dependent is that the determinant of all the $e \times e$ submatrices are zero. These determinants are all polynomials in t. Thus the set of all t such that $(G, \mathbf{p}'(t), \mathbf{q}'(t))$ is dependent is a variety F whose complement, if non-empty, is a dense open set. Since $t = 0$ is in the complement of F, we known that for almost all t, $(G, \mathbf{p}'(t), \mathbf{q}'(t))$ is independent. Choose such a $t_0 \neq 0$. We get an independent realization of G with an extra location for its vertices. This process can be continued until all the vertices are in distinct locations. The proof is thus complete. □

We now give two examples to illustrate the proof of the theorem.

Example 3.5. In Fig. 3.1, the graph at the top has a proper 3T2 decomposition with the edges of T_1 indicated in broken lines, those of T_2 marked in solid lines, while T_3 consists of an isolated vertex e. Now $V_3 = \{a, b, c, d, f\}$, $V_1 = \{e\}$, $V_2 = \varnothing$. The graph in (i) shows the a realization with two locations for the vertices. Note that we show the graph with the vertices in V_i identified and with multiple edges and loops removed. $V_3 \cap T_2$ has three components, $\{a, c, d\}$, $\{b\}$, $\{f\}$. So we can move b and f in the direction of the y-axis to get another independent realization. This is shown in (ii). Now $\langle \{a, c, d\} \rangle \cap T_1$ has three components, a, c, d. So we can move two of these vertices along the x-axis to get an independent realization of the graph as a bar framework as shown in (iii).

Example 3.6. At the top of Fig. 3.2 we show a different 3T2 decomposition of the same graph. The edges of T_1, T_2, T_3 are indicated in thick lines, broken lines and thin lines, respectively. $V_1 = \{b, e, f\}$, $V_2 = \{c\}$, $V_3 = \{a, d\}$. The first realization is shown in (i). $\langle V_i \rangle \cap T_3$ has two components, $\{e, f\}$ and $\{b\}$. So we can move b to another location along the x-axis to get another independent realization; this is shown in (ii). Now $\langle \{e, f\} \rangle \cap T_2$ and $\langle \{a, d\} \rangle \cap T_2$ each has two components. Thus we can move f in the direction $(1, -1)$ and d along the y-axis to a realization of the graph as an independent bar framework as shown in (iii).

Remark 3.7. The condition for the bar framework considered in the above examples to be independent and infinitesimally rigid is well known: *the two triangles abc, and def are not collinear and the three lines ad, be and cf are not concurrent or parallel.* Thus the realizations shown are indeed independent and rigid.

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