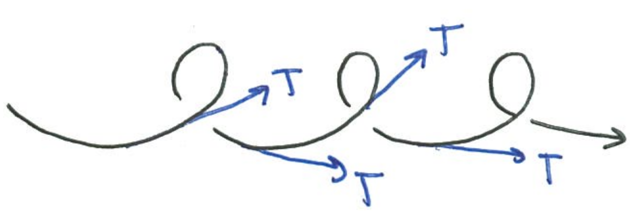
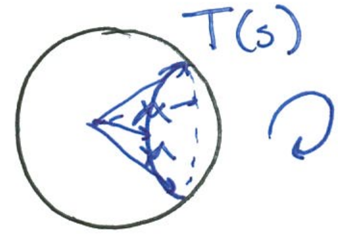
Indicatrices, and some Comparisons.

We now know that κ, τ or κ_1, κ_2 completely determine the geometry of a space curve. This means that we ought to want to decode these functions to understand curve geometry.

Here's a helpful construction:

Definition. The tangent indicatrix of $\gamma(s)$ is the spherical curve $T(s)$.

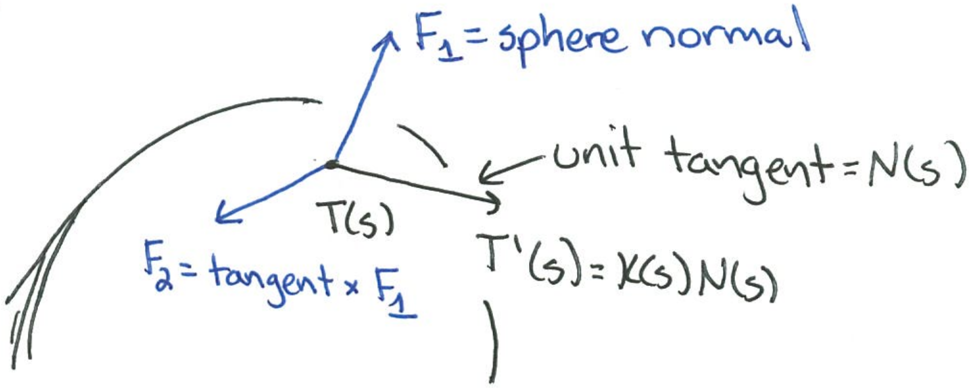




Proposition. $T(s)$ is not arclength-parametrized (unless γ has constant curvature); the speed of $T(s)$ is equal to $\kappa(s)$, the length of $T(s)$ is the total curvature $\int \kappa(s)\,ds$.

Proof. $T'(s) = \kappa(s) N(s)$, so $|T'(s)| = \kappa(s)$, recalling that $\kappa(s) \geq 0$.

If we think of $T(s)$ as a new space curve, it's helpful to frame it by the normal to the sphere.



This is going to lead us into a notational morass, so let's establish conventions now.

Original Curve

$\gamma(s)$ $\kappa(s)$

$T(s)$ $\tau(s)$

$N(s)$ $\kappa_1(s)$

$B(s)$ $\kappa_2(s)$

s = arclength parameter

Tangent indicatrix

$\tilde{\gamma}(\tilde{s})$ $\tilde{\kappa}(\tilde{s})$

$\tilde{T}(\tilde{s})$ $\tilde{\tau}(\tilde{s})$

$\tilde{F}_1(\tilde{s})$ $\tilde{\kappa}_1(\tilde{s})$

$\tilde{F}_2(\tilde{s})$ $\tilde{\kappa}_2(\tilde{s})$

$\tilde{s}$ = arclength parameter.

We have so far

$\tilde{\gamma}(s) = T(s)$ $\quad$ $\tilde{F}_1(s)$ = sphere normal $= T(s)$

$\tilde{T}(s) = N(s)$ $\quad$ $\tilde{F}_2(s) = \tilde{T} \times \tilde{F}_1$.

$|\tilde{\gamma}'(s)| = \kappa(s)$

Proposition. $\tilde{F}_1$ is a Bishop frame, with $\tilde{\kappa}_1 = 1$.

Proof. We need only check that $\tilde{F}_1'$ is parallel to $\tilde{T}$. But

$$\frac{d}{d\tilde{s}} F_1(s(\tilde{s})) = \frac{d}{ds} F_1(s(\tilde{s})) \cdot \frac{d}{d\tilde{s}} s$$

$$= \frac{d}{ds} T(s) \cdot \frac{d}{d\tilde{s}} s = \kappa(s) N(s) \cdot \frac{d}{d\tilde{s}} s$$

$$= \underbrace{\left(\kappa(s)\frac{d}{d\tilde{s}}s\right)}_{\text{scalar}}\tilde{T}(s)$$

In fact, we can compute $\frac{d}{d\tilde{s}}s$ using the fact that $\frac{d}{ds}\tilde{s} = \kappa$. Since $s(\tilde{s})$ and $\tilde{s}(s)$ are inverse functions, this means that $\frac{d}{d\tilde{s}}s = 1/\kappa$ and

$$\frac{d}{d\tilde{s}}F_1 = \tilde{T}(\tilde{s})$$

and so $\tilde{K}_1 = 1$. $\square$

We now compute $\tilde{K}_2$. This is easy if we just realize it's

$$\left\langle \tilde{T}, \frac{d}{d\tilde{s}}\tilde{T}\times\tilde{F}_1\right\rangle = \left\langle \tilde{T}, \left(\frac{d}{d\tilde{s}}\tilde{T}\right)\times\tilde{F}_1\right\rangle + \overset{0}{\cancel{\left\langle \tilde{T}, \tilde{T}\times\frac{d}{ds}\tilde{F}_1\right\rangle}}$$

Now

$$\frac{d}{d\tilde{s}}\tilde{T} = \frac{d}{d\tilde{s}}N(s(\tilde{s})) = \left(-\kappa(s)T(s) + \tau(s)B(s)\right)\frac{d}{d\tilde{s}}s$$

We know

$$\tilde{F}_1(s) = T(s),$$

so $\cancel{K}\ \frac{d}{d\tilde{s}}\tilde{T} \times F_1$ is given by

$$\left[\left(-K(s)\frac{d}{d\tilde{s}}s\right)T(s) + \left(\tau(s)\frac{d}{d\tilde{s}}s\right)B(s)\right] \times T =$$

$$\tau(s)\frac{d}{d\tilde{s}}s\ N$$

and our dot product is

$$= \left\langle N, \tau(s)\frac{d}{d\tilde{s}}s\ N\right\rangle = \tau(s)\frac{d}{d\tilde{s}}s = \frac{\tau(s)}{K(s)}.$$

We now know that

$$\tilde{K}_1 = 1, \quad \tilde{K}_2 = \frac{\tau(s)}{K(s)},$$

so

$$\tilde{K} = \sqrt{K_1^2 + K_2^2} = \sqrt{1 + \frac{\tau^2}{K^2}} = \frac{\sqrt{\tau^2 + K^2}}{K}.$$

It's a little weird to see the κ in the denominator, but it makes sense when you write

$$\text{total curvature of } \tilde{\gamma} = \int \tilde{\kappa}(\tilde{s})\, d\tilde{s}$$

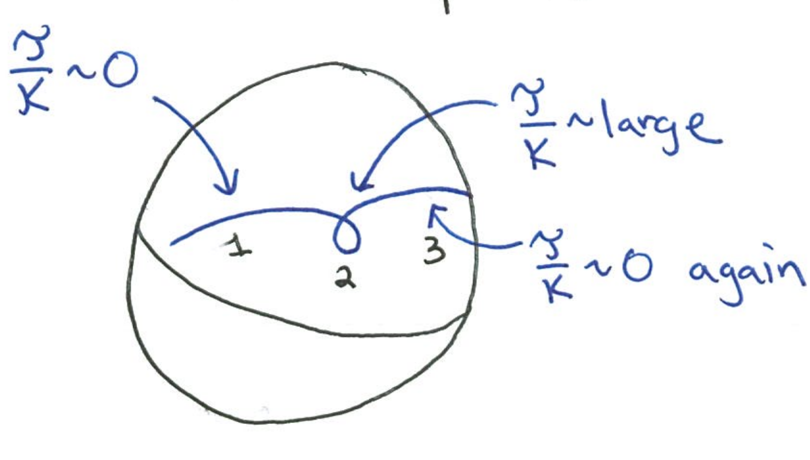
$$= \int \sqrt{\tilde{\kappa}_1^2 + \tilde{\kappa}_2^2}\, d\tilde{s} = \int \frac{\sqrt{\kappa^2+\tau^2}}{\kappa}\, d\tilde{s}$$

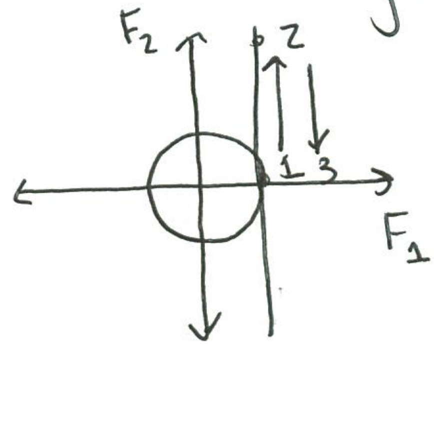
$$= \int \sqrt{\kappa^2+\tau^2}\, \frac{ds}{d\tilde{s}}\, d\tilde{s} = \int \sqrt{\kappa^2+\tau^2}\, ds$$

This is also recognizable as

$$\int |N'(s)|\, ds = \text{length of normal indicatrix.}$$

We can now see qualitatively the relationship between $\mathcal{T}$ and curve geometry





Further, we can see:

Proposition. Any two curves with the same tangent indicatrix have the same total curvature $\int \kappa \, ds$ and same $\int \sqrt{\kappa^2+\tau^2}\, ds$.

Here we mean ~~same~~ "related by a reparametrization" when we say "same tangent indicatrix". Further, at corresponding points, the ratio τ/κ is preserved as well.

Example. Scaling the curve by λ scales κ and τ by $1/\lambda$ and reparametrizes the tangent indicatrix by $1/\lambda$.

a constant factor of $1/\lambda$

We are now able to observe some global features of the tangent indicatrix.

Proposition. A spherical curve γ is the tangent indicatrix of a closed curve $\iff$ γ crosses every plane through the center of the sphere.

Proof. ($\Rightarrow$) Suppose $\gamma = T(s)$ for some closed curve $\alpha(s)$. We know that if α has length L,

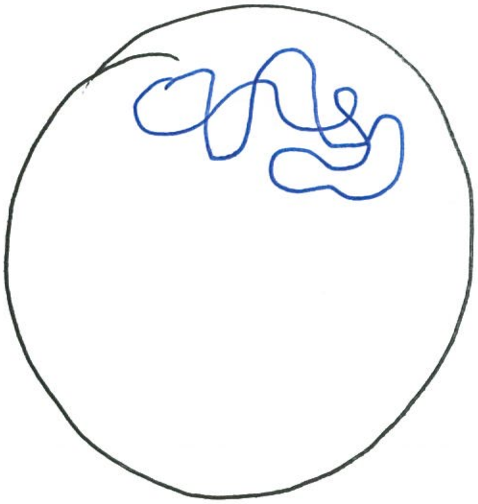
$$\vec{0} = \alpha(L) - \alpha(0) = \int_0^L \alpha'(s)\,ds = \int_0^L T(s)\,ds.$$

Thus $\gamma(s) = T(s)$ has center of mass at $\vec{0}$. Further, for any plane through $\vec{0}$ with normal vector $\vec{n}$,

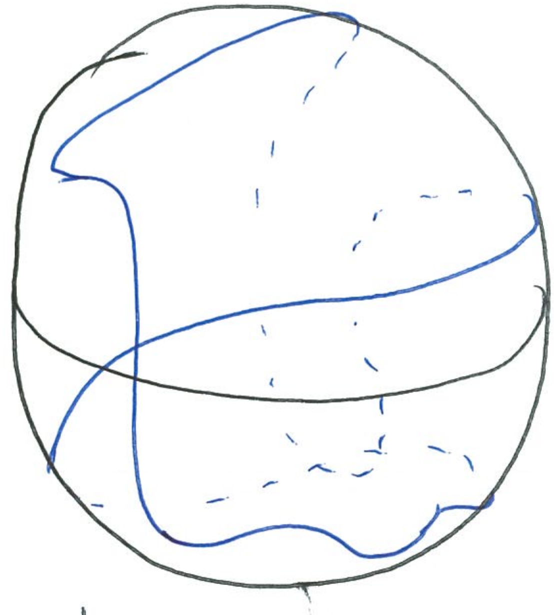
$$0 = \left\langle \vec{n}, \int_0^L T(s)\,ds \right\rangle = \int_0^L \langle \vec{n}, T(s) \rangle\,ds$$

So at some point, $\langle \vec{n}, T(s_*) \rangle = 0$, and T crosses the plane. ($\square$, for $\Rightarrow$)

This lets us classify:



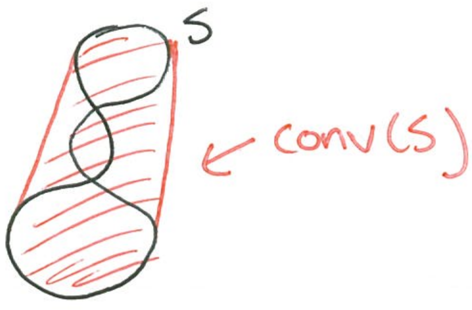
<u>not</u> T(s) for a closed curve



~~could~~ T(s) for some closed curve.

Then we have to prove ($\Leftarrow$).

The <u>convex hull</u> of a set is the intersection of all the halfspaces containing it.



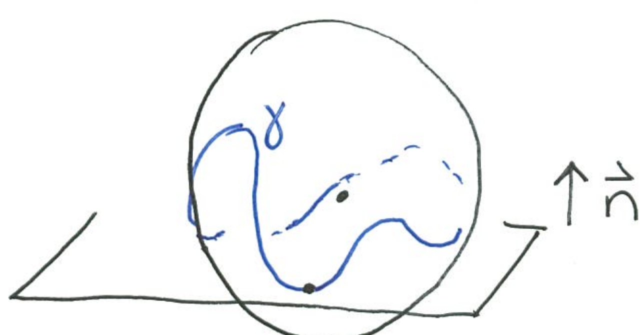
It's a useful theorem that

$$\mathrm{conv}(S) = \left\{ p \,\middle|\, p = \int_S d\mu \right\}$$

where $d\mu \geq 0$ everywhere on S. This is true in ~~a~~ frightening generality ($d\mu$ can contain point masses, or be even weirder!), but we'll only need to know

$$\mathrm{conv}(S) = \left\{ p \,\middle|\, p = \int w(s) \, \gamma(s) \, ds \right\}$$

for some weight function $w(s) \geq 0$, where s is arclength along the spherical curve $\gamma(s)$.



We now show that $\vec{0}$ in $\mathrm{conv}(\gamma)$. Given a halfspace h containing γ with normal $\vec{n}$, slide in the $\vec{n}$ direction until we contact γ.

with the boundary plane. All subsequent planes cut γ until we ~~leave~~ lose contact (forever) at the top of γ.

Since the plane with normal $\vec{n}$ through $\vec{0}$ does cut γ (by hypothesis), if our halfspace $\vec{h}$ contains γ, it contains $\vec{0}$.

Thus $\exists$ some $\omega(s)$ so $\int \gamma(s)\,\omega(s)\,ds = \vec{0}$

Reparametrize by s^* so that $ds^* = \omega(s)\,ds$, and the curve

$$\alpha(s^*) = \int_0^{s^*} \gamma(s^*)\,ds^*$$

has $\alpha'(s^*) = \gamma(s^*)$ and $\int_0^L \alpha'(s^*)\,ds^* = 0$, so $\alpha(s^*)$ is a closed curve with tantrix γ, as desired. $\square$