Surfaces, parametrization, first fundamental form.

We now move from curves to surfaces. As before, we will always deal with parametrized surfaces.

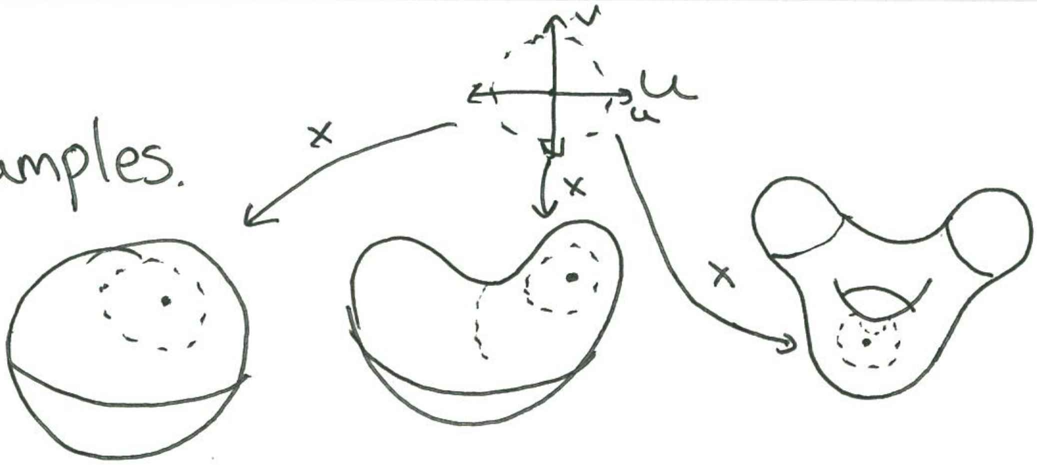
Definition. A regular parametrization of a subset $M \subset \mathbb{R}^3$ is a 1-1 function

$$x : U \to M \quad \text{so} \quad x_u \times x_v \neq \vec{0}$$

for an open set $U \subset \mathbb{R}^2$. A connected subset of $\mathbb{R}^3$ is called a surface if each point has a regularly parametrized neighborhood.

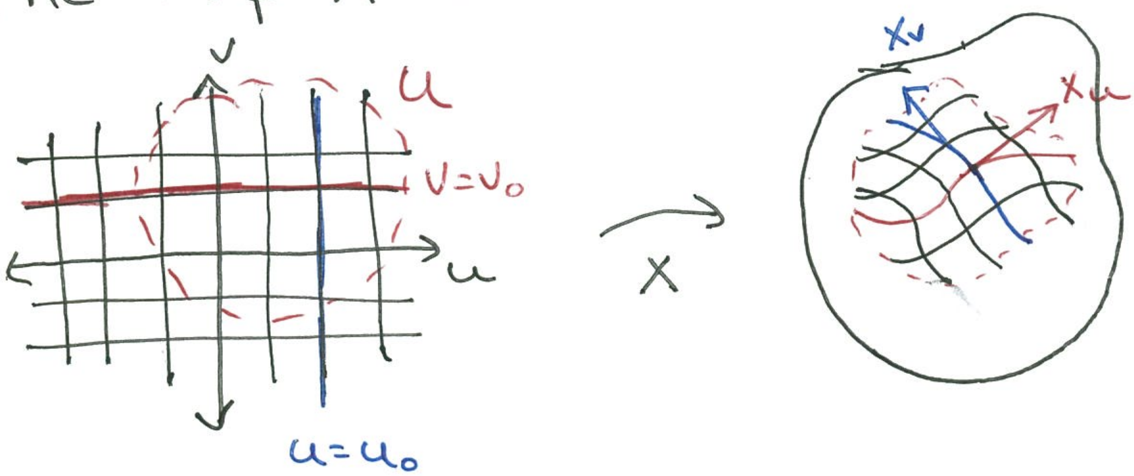
This is already puzzling!

Examples.



The point is that we may not be able to establish a single consistent set of local coordinates (u,v) on the entire surface.

The map x takes curves $x(u_0,v)$, $x(u,v_0)$

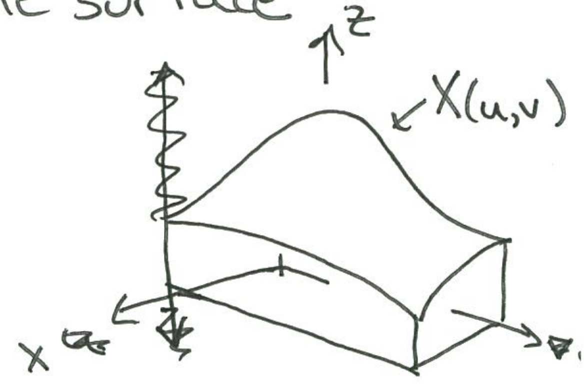


to space curves on M. Their tangents are the partial derivatives x_u, x_v and we require that x_u, x_v span a plane with normal $x_u \times x_v$.

Examples.

The graph of $f: U \to \mathbb{R}$ is the surface

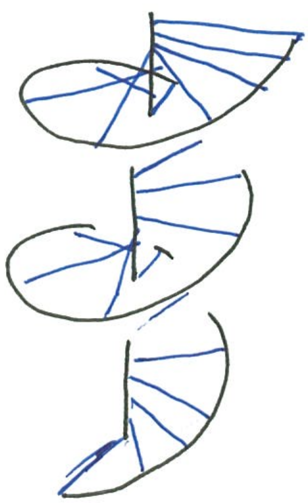
$$X(u,v) = (u, v, f(u,v))$$



We compute

$$X_u \times X_v = (1, 0, f_u) \times (0, 1, f_v)$$
$$= (-f_u, -f_v, 1) \neq 0$$

The helicoid is the surface formed by drawing horizonal rays from the axis of a helix.



$$X(u,v) = (u \cos v, u \sin v, bv)$$

We compute

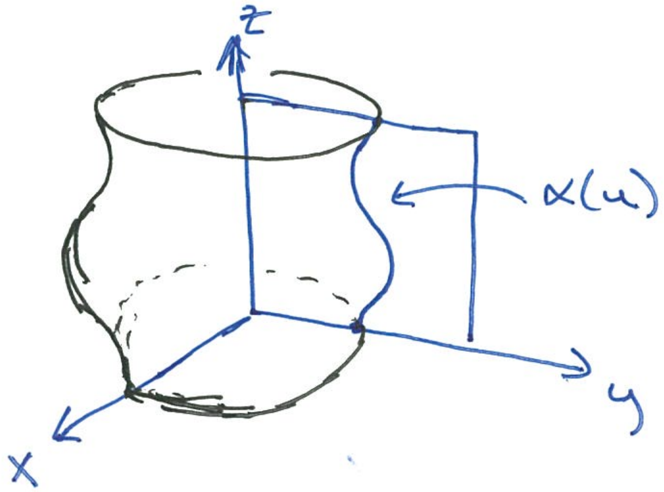
$$X_u \times X_v = (\cos v, \sin v, 0) \times (-u \sin v, u \cos v,$$

$$= (b \sin v, -b \cos v, u) \neq 0$$

aszlong

The surface of revolution given by rotating $\alpha(u) = (0, f(u), g(u))$ around the z-axis is given by

$$X(u,v) = (f(u)\cos v, f(u)\sin v, g(u))$$



We compute

$$X_u \times X_v = (f'(u)\cos v, f'(u)\sin v, g'(u)) \times (-f(u)\sin v, f(u)\cos$$
$$= (-f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f'(u)f(u))$$

Thus

$$|X_u \times X_v| = |f(u)|\sqrt{f'(u)^2 + g'(u)^2} = |f(u)|\,|\alpha'(u)|$$

and the surface is regular away from the z-axis as long as $\alpha(u)$ is a regular curve.

Given a regular curve $\alpha(u)$ and a curve $\beta(u)$ (not through the origin) we can define

$$X(u,v) = \alpha(u) + v\beta(u)$$

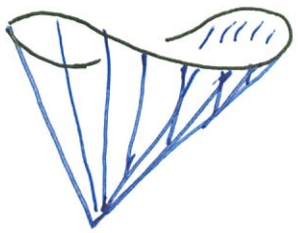
Since the curves $X(u_0,v)$ are straight lines, we call this a <u>ruled</u> surface and β the <u>rulings</u> (α is called the <u>directrix</u>).

$$X_u \times X_v = (\alpha'(u) + v\beta'(u)) \times \beta(u)$$

so this may or may not be regular.

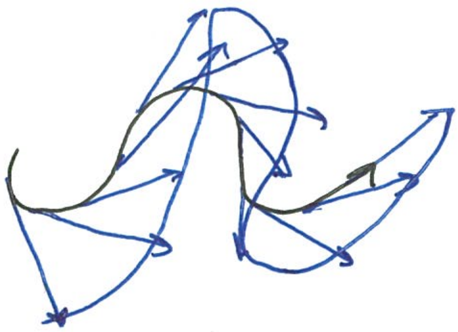
Examples.

$\alpha(u) = \vec{0}$, gives the cone over β



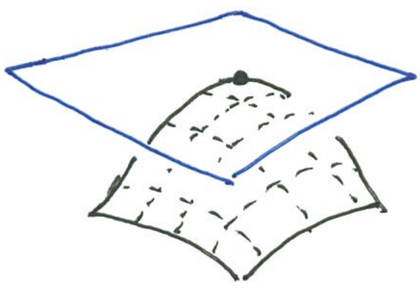
$\alpha(u)$ some curve, $\beta(u) = \alpha'(u)$

This is called the tangent developable.



For curves, we were very concerned with the tangent vector. For surfaces, we have

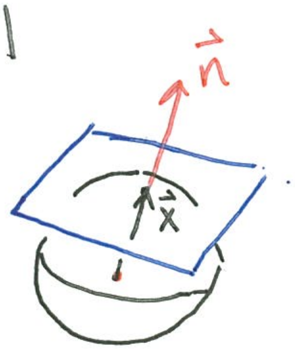
Definition. Let M be a parametrized surface with regular parametrization X, and suppose $P = X(u_0, v_0)$. The tangent plane of M at P is the plane $T_P M$ through P with normal $X_u \times X_v$.



Important Question. Does T_pM depend on the choice of parametrization? Or only on M itself?

Our notation should lead you to guess that different choices of x give you the <u>same</u> plane. This is true.

Definition. The unit normal $\vec{n}$ of M is the vector $\vec{n} = \dfrac{x_u \times x_v}{|x_u \times x_v|}$.

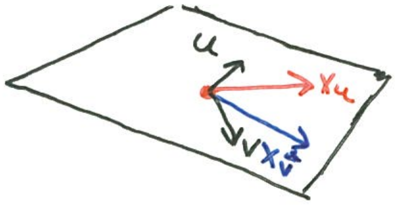


Examples.

1) M is the unit sphere.
$\vec{n} = \vec{x}$.

2) M is the graph of $f(u,v)$.
$$\vec{x}_u \times \vec{x}_v = (-f_u, -f_v, 1), \quad \vec{n} = \frac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}}.$$

We now introduce a way to do measurements on a surface M.



Given vectors u, v in T_pM, we can write them in the x_u, x_v basis as

$$u = a x_u + b x_v$$
$$v = c x_u + d x_v$$

Definition. The first fundamental form I_p is the symmetric 2×2 matrix so that

$$\langle (a,b), I_p(c,d) \rangle_{\mathbb{R}^2} = \langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^3}$$

Proposition. $I_p = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ where $E = \langle \vec{x}_u, \vec{x}_u \rangle_{\mathbb{R}^3}$,

$F = \langle \vec{x}_u, \vec{x}_v \rangle_{\mathbb{R}^3}$, $G = \langle \vec{x}_v, \vec{x}_v \rangle_{\mathbb{R}^3}$.

Proof. On one hand,

$$\left\langle (a,b), \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \right\rangle_{\mathbb{R}^2} = \left\langle (a,b), (Ec+Fd, Fc+Gd) \right\rangle_{\mathbb{R}^2}$$

$$= Eac + Fad + Fbc + Gbd$$

$$= Eac + F(ad+bc) + Gbd$$

On the other,

$$\langle \vec{u}, \vec{v} \rangle_{\mathbb{R}^3} = \langle a\vec{x}_u + b\vec{x}_v, c\vec{x}_u + d\vec{x}_v \rangle_{\mathbb{R}^3}$$

$$= ac \langle \vec{x}_u, \vec{x}_u \rangle_{\mathbb{R}^3} + (ad+bc) \langle \vec{x}_u, \vec{x}_v \rangle$$
$$+ bd \langle \vec{x}_v, \vec{x}_v \rangle.$$

Since this has to work for all a, b, c, d, it's easy to conclude that E, F, G are as given before. $\square$

A symmetric matrix used in this way is called a <u>quadratic</u> <u>form</u>. ~~The operation~~ It defines a <u>new</u> inner product on $\mathbb{R}^2$

given by

$$I_p(\vec{w}, \vec{z}) = \left\langle \vec{w}, \begin{bmatrix} E & F \\ F & G \end{bmatrix} \vec{z} \right\rangle_{\mathbb{R}^2}$$

$$= \langle \vec{w}, \vec{z} \rangle_{I_p}$$

note notation!

and a new norm on $\mathbb{R}^2$:

$$\| \vec{w} \|^2_{I_p} = I_p(\vec{w}, \vec{w}).$$

We now introduce a deep idea.

Definition. Surfaces M and M^* are <u>locally isometric</u> if ~~they~~ for each $p \in M$ there is a $p^* \in M$ and regular parametrizations $X : U \to M$ and $X^* : U \to M^*$ covering p, p^* so that for <u>all</u> (u,v) in ~~4~~ U we have

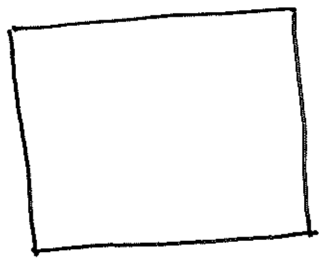
$$I_{X(u,v)} = I_{X^*(u,v)}$$

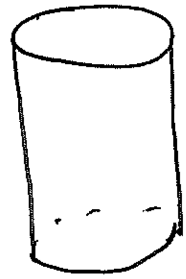
or more specifically,

$$E = \langle X_u, X_u \rangle = \langle X^*_u, X^*_u \rangle = E^*$$
$$F = \langle X_u, X_v \rangle = \langle X^*_u, X^*_v \rangle = F^*$$
$$G = \langle X_v, X_v \rangle = \langle X^*_v, X^*_v \rangle = G^*$$

Note that we did <u>not</u> ask that M and M^* be related by a rigid motion of $\mathbb{R}^3$, though that would suffice!

Example.





$$X(u,v) = (u,v,0)$$

$$X_u = (1,0,0)$$
$$X_v = (0,1,0)$$

$$E = \langle X_u, X_u \rangle = 1$$
$$F = \langle X_u, X_v \rangle = 0$$
$$G = \langle X_v, X_v \rangle = 1$$

$$X^*(u,v) = (\cos u, \sin u, v)$$

$$X_u^* = (-\sin u, \cos u, 0)$$
$$X_v^* = (0,0,1)$$

$$E^* = \langle X_u^*, X_u^* \rangle = \sin^2 u + \cos^2 u = 1$$
$$F^* = \langle X_u^*, X_v^* \rangle = 0$$
$$G^* = \langle X_v^*, X_v^* \rangle = 1$$

So these are locally isometric!