# Geodesics as Straightest Paths

We start with a lemma:

**Lemma.** If $X$ and $Y$ are parallel vector fields on $\alpha(t)$ then $\|X\|$, $\|Y\|$ and the angle between $X$ and $Y$ are constant in $t$.

**Proof.** Consider

$$\frac{d}{dt}\langle X(\alpha(t)), Y(\alpha(t))\rangle = \langle X', Y\rangle + \langle X, Y'\rangle$$

Now $\nabla_{\alpha'(t)} X(\alpha(t)) = \frac{d}{dt}X(\alpha(t)) - \langle \frac{d}{dt}X(\alpha(t)), \vec{n}\rangle \vec{n}$, but since $Y \in T_pM$, this means

$$\langle \tfrac{d}{dt}X, Y\rangle = \langle \nabla_{\alpha'(t)}X, Y\rangle$$

Thus our original

$$\frac{d}{dt}\langle X, Y\rangle = \overbrace{\langle \nabla_{\alpha'(t)}X, Y\rangle}^{0,\text{ since } X \text{ parallel}} + \overbrace{\langle X, \nabla_{\alpha'(t)}Y\rangle}^{0,\text{ since } Y \text{ parallel.}}$$

$$= 0 + 0 = 0. \qquad \square$$

We can now define

**Definition.** A curve $\alpha(t) \in M$ is a geodesic if $\alpha'(t)$ is parallel: $\nabla_{\alpha'(t)} \alpha'(t) = 0$.

By the lemma, this means $|\alpha'(t)|$ is constant, so $\alpha(t)$ is parametrized by a constant multiple of arclength.

We can write $\cancel{\alpha(t)}$ $\kappa N$ in the Darboux frame $\{T, \vec{n} \times T, \vec{n}\}$ (where $\vec{n}$ is the surface normal) as

$$\kappa N = \underbrace{\langle \kappa N, \vec{n} \times T \rangle}_{\text{geodesic curvature } \kappa_g} \vec{n} \times T + \underbrace{\langle \kappa N, \vec{n} \rangle}_{\text{normal curvature } \kappa_{\vec{n}}} \vec{n}$$

**Lemma.** $\alpha(t)$ is a geodesic $\Leftrightarrow \kappa_g \equiv 0$.

We see $\alpha''(t) = \kappa N$, so

$$\nabla_{\alpha'(t)} \alpha'(t) = \kappa N - \langle \kappa N, \vec{n} \rangle \vec{n}.$$

This is 0 exactly when $\kappa_g$ vanishes.

We can now simplify our earlier equations for parallel transport by observing that $a(t) = u'(t)$, $b(t) = v'(t)$. We get

$$u''(t) + \Gamma^u_{uu}\, u'(t)^2 + 2\Gamma^u_{uv}\, u'(t)v'(t) + \Gamma^u_{vv}\, v'(t)^2 = 0$$

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Proposition. Given $p \in M$ and $\vec{v} \in T_pM \neq \vec{0}$, there is an $(-\epsilon, \epsilon)$ and a unique geodesic $\alpha(t)$ with $\alpha(0) = p$ and $\alpha'(0) = v$.

This is again a consequence of our theorem on solutions of ODEs.

Example. Let $X(u,v) = (u,v,0)$ parametrize the plane. Since $X_{uu} = X_{vv} = 0$, all Christoffel symbols are $0$ and the geodesic equations are $u'' = 0$, $v'' = 0$.

Example. Revisiting the sphere

$$X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

we recall that all Christoffel symbols were zero except $\Gamma_{uv}^{v} = \cot u$, $\Gamma_{vv}^{u} = -\sin u \cos u$.
So we get

$$u''(t) - \sin u(t) \cos u(t)\, v'(t)^2 = 0$$
$$v''(t) + 2\cot u(t)\, u'(t)\, v'(t) = 0.$$

One set of solutions is $v(t) = v_0$, $u(t) = t$.
These give great circles through the north pole!

We'll have to work for everything else.
Now separating variables,

$$\frac{v''(t)}{v'(t)} = -2\cot u(t)\, u'(t) = \frac{-2(\cos u)\, u'}{\sin u}$$

so (integrating)

$$\ln v'(t) = -2 \ln \sin u + C$$

$$= -2\ln(C\sin u) \qquad \text{(c changed!)}$$

$$= \ln\left(\frac{c}{\sin^2 u}\right) \qquad \text{(c changed again!)}$$

so

$$V'(t) = \frac{c}{\sin^2 u} = c\csc^2 u$$

Now if we plug this into the first equation,

$$u'' - \frac{\cancel{\sin u}\cos u\, c^2}{\sin^{\cancel{4}3} u} = 0$$

so

$$u'' = \frac{c^2\cos u}{\sin^3 u}$$

Here's a neat trick. We'd like to integrate both sides, ~~but~~ ~~do~~ wrt $u$, but don't have a $u'(t)$ on either. Why not add one? Multiplying through by $u'(t)$, we get

$$u'' \cdot u' = \frac{c^2 \cos u}{\sin^3 u} u'$$

or (integrating w.r.t $u$)

$$\frac{(u')^2}{2} = \frac{c^2}{-2} \sin^{-2} u + C$$

Simplifying, we get

$$(u')^2 = C - \frac{c^2}{\sin^2 u} \qquad \text{(C changed)}$$

or

$$u' = \pm \sqrt{C - \frac{c^2}{\sin^2 u}}$$

We now write

$$\frac{dv}{du} = \frac{v'(t)}{u'(t)} = \pm \frac{c \csc^2 u}{\sqrt{C^2 - c^2 \csc^2 u}}$$

Now this is more hopeful than it looks!

$\sin^2 u + \cos^2 u = 1$, so $1 + \cot^2 u = \csc^2 u$

Also,

$$\frac{d}{du}\cot u = -\csc^2 u.$$

So first, we have

$$dv = \pm \frac{c\csc^2 u\, du}{\sqrt{C^2 - c^2(1+\cot^2 u)}}$$

$$= \pm \frac{c\csc^2 u\, du}{\sqrt{(C^2-c^2) - c^2\cot^2 u}}$$

Now let's make a clever change of variables:

$$c\cot u = \sqrt{C^2-c^2}\sin\omega$$

$$-c\csc^2 u\, du = \sqrt{C^2-c^2}\cos\omega\, d\omega$$

Now

$$c^2\cot^2 u = (C^2-c^2)\sin^2\omega,$$

so

$$\sqrt{(C^2-c^2) - c^2\cot^2 u} = \sqrt{(C^2-c^2) - (C^2-c^2)\sin^2\omega}$$

$$= \sqrt{C^2-c^2}\sqrt{1-\sin^2\omega} = \sqrt{C^2-c^2}\cos\omega.$$

Thus

$$dv = \pm \frac{\csc^2 u \, du}{\sqrt{(C^2-c^2) - c^2 \cot^2 u}}$$

$$= \cancel{\pm}\mp \frac{\sqrt{C^2-c^2}\cos\omega \, d\omega}{\sqrt{C^2-c^2}\cos\omega} = \mp d\omega.$$

and (amazingly) we have

$$\omega = \pm v + a$$

for some constant $a$. Thus

$$c \cot u = \sqrt{C^2-c^2}\sin\omega$$

$$= \sqrt{C^2-c^2}\sin(\pm v + a)$$

$$= \sqrt{C^2-c^2}(\sin a \cos v \pm \cos a \sin v)$$

or

$$c \cos u = \sqrt{C^2-c^2}\sin u (A\cos v + B\sin v).$$

Now you've undoubtedly forgotten what $u$ and $v$ mean at this point, but recall that

$$X(u,v) = (\sin u \cos v, \sin u \sin v, \cos u)$$

so we can call these $X, Y, Z$ and observe the equation is

$$cZ = (A\sqrt{C^2-c^2})\,X + (B\sqrt{C^2-c^2})\,Y,$$

or the equation implies $\alpha$ lies on a plane through the origin (with constant equation) and hence $\alpha(t)$ is a great circle.

Conclusion. There are no other ~~great~~ ~~circle~~ geodesics on $S^2$!

Of course, you could probably have gotten that from uniqueness, but this was more fun. ☺