Alternatively, since $\tan(\theta/2) = e^t$, we have

$$\sin\theta = 2\sin(\theta/2)\cos(\theta/2) = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^t+e^{-t}} = \operatorname{sech} t$$
$$\cos\theta = \cos^2(\theta/2) - \sin^2(\theta/2) = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t}-e^t}{e^t+e^{-t}} = -\tanh t,$$

and so we can parametrize the tractrix instead by

$$\boldsymbol{\beta}(t) = \big(t - \tanh t, \operatorname{sech} t\big), \quad t \geq 0. \qquad \nabla$$

The fundamental concept underlying the geometry of curves is the arclength of a parametrized curve.

Definition. If $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ is a parametrized curve, then for any $a \leq t \leq b$, we define its *arclength* from a to t to be $s(t) = \displaystyle\int_a^t \|\boldsymbol{\alpha}'(u)\|\,du$. That is, the distance a particle travels—the arclength of its trajectory—is the integral of its speed.

An alternative approach is to start with the following

Definition. Let $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ be a (continuous) parametrized curve. Given a partition $\mathcal{P} = \{a = t_0 < t_1 < \cdots < t_k = b\}$ of the interval $[a,b]$, let

$$\ell(\boldsymbol{\alpha}, \mathcal{P}) = \sum_{i=1}^{k} \|\boldsymbol{\alpha}(t_i) - \boldsymbol{\alpha}(t_{i-1})\|.$$

That is, $\ell(\boldsymbol{\alpha}, \mathcal{P})$ is the length of the inscribed polygon with vertices at $\boldsymbol{\alpha}(t_i)$, $i = 0, \ldots, k$, as indicated in

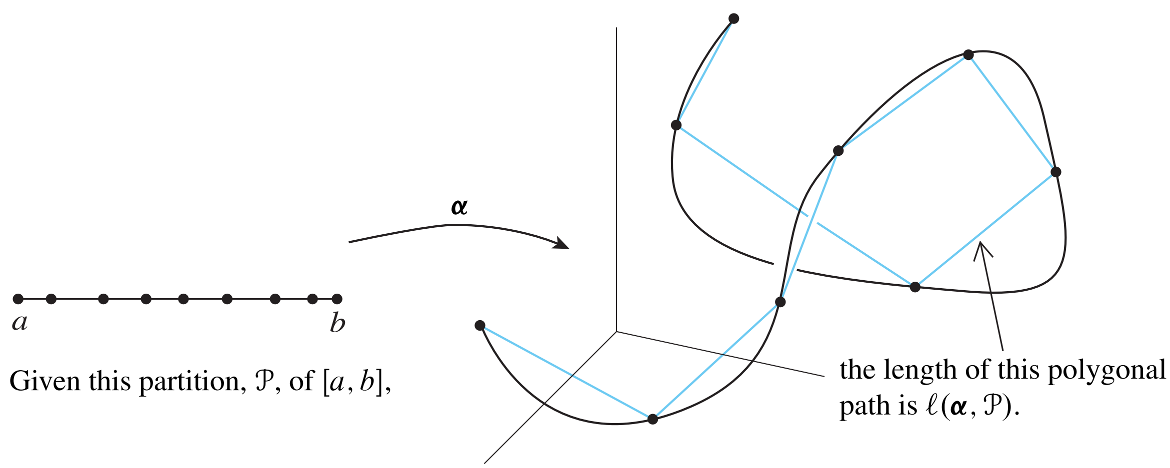

FIGURE 1.10

Figure 1.10. We define the *arclength* of $\boldsymbol{\alpha}$ to be

$$\text{length}(\boldsymbol{\alpha}) = \sup\{\ell(\boldsymbol{\alpha}, \mathcal{P}) : \mathcal{P} \text{ a partition of } [a,b]\},$$

provided the set of polygonal lengths is bounded above.

Now, using this definition, we can *prove* that the distance a particle travels is the integral of its speed. We will need to use the result of Exercise A.2.4.

Proposition 1.1. *Let* $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ *be a piecewise-*$\mathcal{C}^1$ *parametrized curve. Then*

$$\text{length}(\boldsymbol{\alpha}) = \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt.$$

Proof. For any partition $\mathcal{P}$ of $[a,b]$, we have

$$\ell(\boldsymbol{\alpha},\mathcal{P}) = \sum_{i=1}^{k} \|\boldsymbol{\alpha}(t_i) - \boldsymbol{\alpha}(t_{i-1})\| = \sum_{i=1}^{k} \left\| \int_{t_{i-1}}^{t_i} \boldsymbol{\alpha}'(t)\,dt \right\| \le \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \|\boldsymbol{\alpha}'(t)\|\,dt = \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt,$$

so $\text{length}(\boldsymbol{\alpha}) \le \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt$. The corresponding inequality holds on any interval.

Now, for $a \le t \le b$, define $s(t)$ to be the arclength of the curve $\boldsymbol{\alpha}$ on the interval $[a,t]$. Then for $h > 0$ we have

$$\frac{\|\boldsymbol{\alpha}(t+h) - \boldsymbol{\alpha}(t)\|}{h} \le \frac{s(t+h) - s(t)}{h} \le \frac{1}{h}\int_t^{t+h} \|\boldsymbol{\alpha}'(u)\|\,du,$$

since $s(t+h) - s(t)$ is the arclength of the curve $\boldsymbol{\alpha}$ on the interval $[t, t+h]$. (See Exercise 8 for the first inequality and the first paragraph for the second.) Now

$$\lim_{h\to 0^+} \frac{\|\boldsymbol{\alpha}(t+h) - \boldsymbol{\alpha}(t)\|}{h} = \|\boldsymbol{\alpha}'(t)\| = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \|\boldsymbol{\alpha}'(u)\|\,du.$$

Therefore, by the squeeze principle,

$$\lim_{h\to 0^+} \frac{s(t+h) - s(t)}{h} = \|\boldsymbol{\alpha}'(t)\|.$$

A similar argument works for $h < 0$, and we conclude that $s'(t) = \|\boldsymbol{\alpha}'(t)\|$. Therefore,

$$s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)\|\,du, \quad a \le t \le b,$$

and, in particular, $s(b) = \text{length}(\boldsymbol{\alpha}) = \int_a^b \|\boldsymbol{\alpha}'(t)\|\,dt$, as desired. □

If $\|\boldsymbol{\alpha}'(t)\| = 1$ for all $t \in [a,b]$, i.e., $\boldsymbol{\alpha}$ always has speed 1, then $s(t) = t - a$. We say the curve $\boldsymbol{\alpha}$ is *parametrized by arclength* if $s(t) = t$ for all t. In this event, we usually use the parameter $s \in [0, L]$ and write $\boldsymbol{\alpha}(s)$.

Example 3. (a) Let $\boldsymbol{\alpha}(t) = \left(\frac{1}{3}(1+t)^{3/2}, \frac{1}{3}(1-t)^{3/2}, \frac{1}{\sqrt{2}}t\right)$, $t \in (-1,1)$. Then we have $\boldsymbol{\alpha}'(t) = \left(\frac{1}{2}(1+t)^{1/2}, -\frac{1}{2}(1-t)^{1/2}, \frac{1}{\sqrt{2}}\right)$, and $\|\boldsymbol{\alpha}'(t)\| = 1$ for all t. Thus, $\boldsymbol{\alpha}$ always has speed 1.

(b) The standard parametrization of the circle of radius a is $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t)$, $t \in [0, 2\pi]$, so $\boldsymbol{\alpha}'(t) = (-a\sin t, a\cos t)$ and $\|\boldsymbol{\alpha}'(t)\| = a$. It is easy to see from the chain rule that if we reparametrize the curve by $\boldsymbol{\beta}(s) = (a\cos(s/a), a\sin(s/a))$, $s \in [0, 2\pi a]$, then $\boldsymbol{\beta}'(s) = (-\sin(s/a), \cos(s/a))$ and $\|\boldsymbol{\beta}'(s)\| = 1$ for all s. Thus, the curve $\boldsymbol{\beta}$ is parametrized by arclength. ▽

An important observation from a theoretical standpoint is that any regular parametrized curve can be *re*parametrized by arclength. For if $\boldsymbol{\alpha}$ is regular, the arclength function $s(t) = \int_a^t \|\boldsymbol{\alpha}'(u)\| du$ is an increasing differentiable function (since $s'(t) = \|\boldsymbol{\alpha}'(t)\| > 0$ for all t), and therefore has a differentiable inverse function $t = t(s)$. Then we can consider the parametrization

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(t(s)).$$

Note that the chain rule tells us that

$$\boldsymbol{\beta}'(s) = \boldsymbol{\alpha}'(t(s))t'(s) = \boldsymbol{\alpha}'(t(s))/s'(t(s)) = \boldsymbol{\alpha}'(t(s))/\|\boldsymbol{\alpha}'(t(s))\|$$

is everywhere a unit vector; in other words, $\boldsymbol{\beta}$ moves with speed 1.

EXERCISES 1.1

*1. Parametrize the unit circle (less the point $(-1, 0)$) by the length t indicated in Figure 1.11.

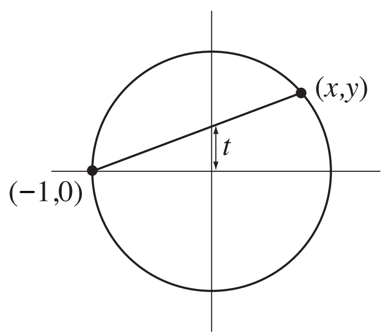

FIGURE 1.11

♯2. Consider the helix $\boldsymbol{\alpha}(t) = (a \cos t, a \sin t, bt)$. Calculate $\boldsymbol{\alpha}'(t)$, $\|\boldsymbol{\alpha}'(t)\|$, and reparametrize $\boldsymbol{\alpha}$ by arclength.

3. Let $\boldsymbol{\alpha}(t) = \left(\frac{1}{\sqrt{3}} \cos t + \frac{1}{\sqrt{2}} \sin t, \frac{1}{\sqrt{3}} \cos t, \frac{1}{\sqrt{3}} \cos t - \frac{1}{\sqrt{2}} \sin t\right)$. Calculate $\boldsymbol{\alpha}'(t)$, $\|\boldsymbol{\alpha}'(t)\|$, and reparametrize $\boldsymbol{\alpha}$ by arclength.

*4. Parametrize the graph $y = f(x)$, $a \le x \le b$, and show that its arclength is given by the traditional formula

$$\text{length} = \int_a^b \sqrt{1 + \left(f'(x)\right)^2}\, dx.$$

5. a. Show that the arclength of the catenary $\boldsymbol{\alpha}(t) = (t, \cosh t)$ for $0 \le t \le b$ is $\sinh b$.
   b. Reparametrize the catenary by arclength. (Hint: Find the inverse of sinh by using the quadratic formula.)

*6. Consider the curve $\boldsymbol{\alpha}(t) = (e^t, e^{-t}, \sqrt{2}t)$. Calculate $\boldsymbol{\alpha}'(t)$, $\|\boldsymbol{\alpha}'(t)\|$, and reparametrize $\boldsymbol{\alpha}$ by arclength, starting at $t = 0$.