CHAPTER 2

Surfaces: Local Theory

1. Parametrized Surfaces and the First Fundamental Form

Let U be an open set in $\mathbb{R}^2$. A function $\mathbf{f}\colon U \to \mathbb{R}^m$ (for us, $m = 1$ and 3 will be most common) is called $\mathcal{C}^1$ if $\mathbf{f}$ and its partial derivatives $\dfrac{\partial \mathbf{f}}{\partial u}$ and $\dfrac{\partial \mathbf{f}}{\partial v}$ are all continuous. We will ordinarily use (u, v) as coordinates in our parameter space, and (x, y, z) as coordinates in $\mathbb{R}^3$. Similarly, for any $k \geq 2$, we say $\mathbf{f}$ is $\mathcal{C}^k$ if all its partial derivatives of order up to k exist and are continuous. We say $\mathbf{f}$ is *smooth* if $\mathbf{f}$ is $\mathcal{C}^k$ for every positive integer k. We will henceforth assume all our functions are $\mathcal{C}^k$ for $k \geq 3$. One of the crucial results for differential geometry is that if $\mathbf{f}$ is $\mathcal{C}^2$, then $\dfrac{\partial^2 \mathbf{f}}{\partial u \partial v} = \dfrac{\partial^2 \mathbf{f}}{\partial v \partial u}$ (and similarly for higher-order derivatives).

Notation: We will often also use subscripts to indicate partial derivatives, as follows:

$$
\begin{array}{lcc}
\mathbf{f}_u & \leftrightarrow & \dfrac{\partial \mathbf{f}}{\partial u} \\[2ex]
\mathbf{f}_v & \leftrightarrow & \dfrac{\partial \mathbf{f}}{\partial v} \\[2ex]
\mathbf{f}_{uu} & \leftrightarrow & \dfrac{\partial^2 \mathbf{f}}{\partial u^2} \\[2ex]
\mathbf{f}_{uv} = (\mathbf{f}_u)_v & \leftrightarrow & \dfrac{\partial^2 \mathbf{f}}{\partial v \partial u}
\end{array}
$$

Definition. A *regular parametrization* of a subset $M \subset \mathbb{R}^3$ is a ($\mathcal{C}^3$) one-to-one function

$$\mathbf{x}\colon U \to M \subset \mathbb{R}^3 \qquad \text{so that} \qquad \mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$$

for some open set $U \subset \mathbb{R}^2$.[1] A connected subset $M \subset \mathbb{R}^3$ is called a *surface* if each point has a neighborhood that is regularly parametrized.

We might consider the curves on M obtained by fixing $v = v_0$ and varying u, called a u-curve, and obtained by fixing $u = u_0$ and varying v, called a v-curve; these are depicted in Figure 1.1. At the point $P = \mathbf{x}(u_0, v_0)$, we see that $\mathbf{x}_u(u_0, v_0)$ is tangent to the u-curve and $\mathbf{x}_v(u_0, v_0)$ is tangent to the v-curve. We are requiring that these vectors span a plane, whose normal vector is given by $\mathbf{x}_u \times \mathbf{x}_v$.

Example 1. We give some basic examples of parametrized surfaces. Note that our parameters do not necessarily range over an open set of values.

(a) The graph of a function $f\colon U \to \mathbb{R}$, $z = f(x, y)$, is parametrized by $\mathbf{x}(u, v) = (u, v, f(u, v))$. Note that $\mathbf{x}_u \times \mathbf{x}_v = (-f_u, -f_v, 1) \neq \mathbf{0}$, so this is always a regular parametrization.

[1] For technical reasons with which we shall not concern ourselves in this course, we should also require that the inverse function $\mathbf{x}^{-1}\colon \mathbf{x}(U) \to U$ be continuous. We shall also often be sloppy and use subsets U that are not quite open. The interested reader can easily repair things by adding some companion parametrizations.

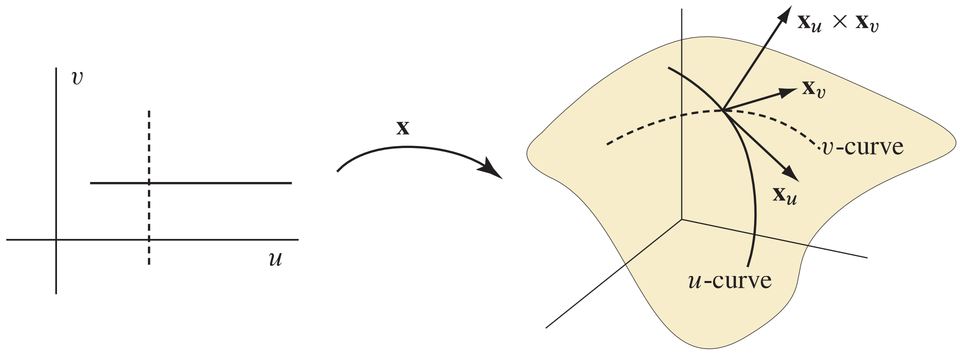

FIGURE 1.1

(b) The *helicoid*, as shown in Figure 1.2, is the surface formed by drawing horizontal rays from the axis

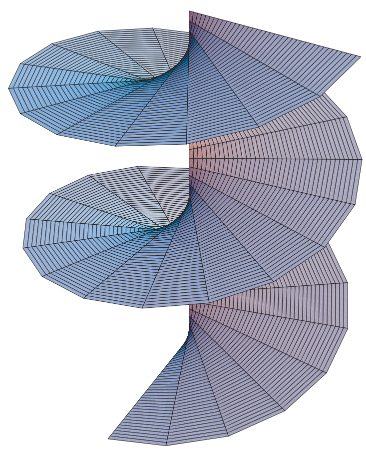

FIGURE 1.2

of the helix $\boldsymbol{\alpha}(t) = (\cos t, \sin t, bt)$ to points on the helix:

$$\mathbf{x}(u, v) = (u \cos v, u \sin v, bv), \qquad u > 0, \ v \in \mathbb{R}.$$

Note that $\mathbf{x}_u \times \mathbf{x}_v = (b \sin v, -b \cos v, u) \neq \mathbf{0}$. The u-curves are rays and the v-curves are helices.

(c) The *torus* (surface of a doughnut) is formed by rotating a circle of radius b about a circle of radius $a > b$ lying in an orthogonal plane, as pictured in Figure 1.3. The regular parametrization is given

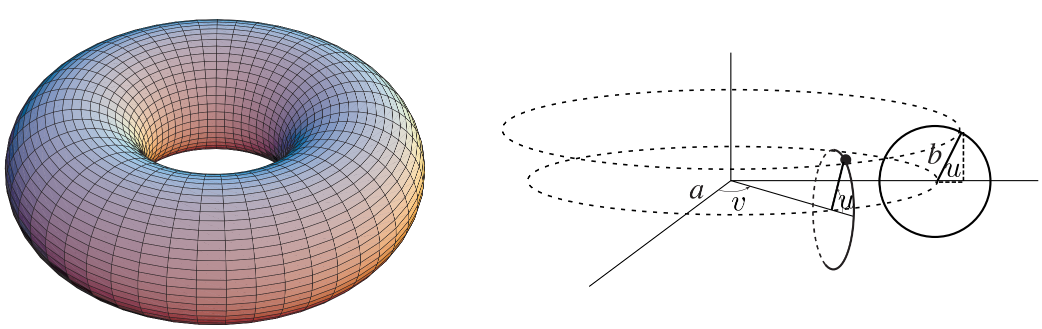

FIGURE 1.3

by

$$\mathbf{x}(u,v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u), \qquad 0 \le u, v < 2\pi.$$

Then $\mathbf{x}_u \times \mathbf{x}_v = -b(a + b\cos u)\big(\cos u\cos v, \cos u\sin v, \sin u\big)$, which is never $\mathbf{0}$.

(d) The standard parametrization of the unit sphere Σ is given by spherical coordinates $(\phi,\theta) \leftrightarrow (u,v)$:

$$\mathbf{x}(u,v) = (\sin u\cos v, \sin u\sin v, \cos u), \qquad 0 < u < \pi,\ 0 \le v < 2\pi.$$

Since $\mathbf{x}_u \times \mathbf{x}_v = \sin u(\sin u\cos v, \sin u\sin v, \cos u) = (\sin u)\mathbf{x}(u,v)$, the parametrization is regular away from $u = 0, \pi$, which we've excluded anyhow because $\mathbf{x}$ fails to be one-to-one at such points. The u-curves are the so-called lines of longitude and the v-curves are the lines of latitude on the sphere.

(e) Another interesting parametrization of the sphere is given by *stereographic projection*. (Cf. Exercise 1.1.1.) We parametrize the unit sphere less the north pole $(0,0,1)$ by the xy-plane, assigning to each

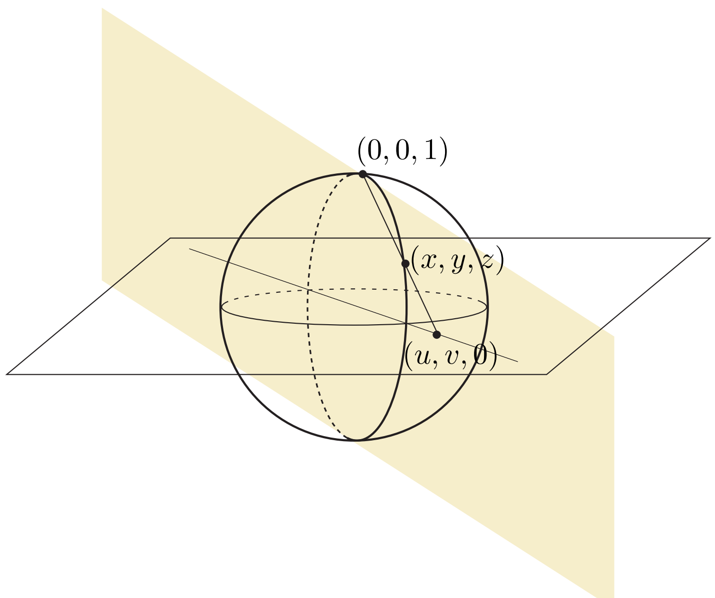

FIGURE 1.4

(u,v) the point $(\neq (0,0,1))$ where the line through $(0,0,1)$ and $(u,v,0)$ intersects the unit sphere, as pictured in Figure 1.4. We leave it to the reader to derive the following formula in Exercise 1:

$$\mathbf{x}(u,v) = \left(\frac{2u}{u^2+v^2+1}, \frac{2v}{u^2+v^2+1}, \frac{u^2+v^2-1}{u^2+v^2+1}\right). \qquad \nabla$$

For our last examples, we give two general classes of surfaces that will appear throughout our work.

Example 2. Let $I \subset \mathbb{R}$ be an interval, and let $\boldsymbol{\alpha}(u) = (0, f(u), g(u))$, $u \in I$, be a regular parametrized plane curve[2] with $f > 0$. Then the *surface of revolution* obtained by rotating $\boldsymbol{\alpha}$ about the z-axis is parametrized by

$$\mathbf{x}(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big), \quad u \in I,\ 0 \le v < 2\pi.$$

[2]Throughout, we assume regular parametrized curves to be one-to-one.

Note that $\mathbf{x}_u \times \mathbf{x}_v = f(u)\big(-g'(u)\cos v, -g'(u)\sin v, f'(u)\big)$, so this is a regular parametrization. The u-curves are often called *profile curves* or *meridians*; these are copies of $\boldsymbol{\alpha}$ rotated an angle v around the z-axis. The v-curves are circles, called *parallels*. ∇

Example 3. Let $I \subset \mathbb{R}$ be an interval, let $\boldsymbol{\alpha}\colon I \to \mathbb{R}^3$ be a regular parametrized curve, and let $\boldsymbol{\beta}\colon I \to \mathbb{R}^3$ be an arbitrary smooth function with $\boldsymbol{\beta}(u) \neq \mathbf{0}$ for all $u \in I$. We define a parametrized surface by

$$\mathbf{x}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u), \quad u \in I,\ v \in \mathbb{R}.$$

This is called a *ruled surface* with *rulings* $\boldsymbol{\beta}(u)$ and *directrix* $\boldsymbol{\alpha}$. It is easy to check that $\mathbf{x}_u \times \mathbf{x}_v = (\boldsymbol{\alpha}'(u) + v\boldsymbol{\beta}'(u)) \times \boldsymbol{\beta}(u)$, which may or may not be everywhere nonzero.

As particular examples, we have the helicoid (see Figure 1.2) and the following (see Figure 1.5):

(1) Cylinder: Here $\boldsymbol{\beta}$ is a constant vector, and the surface is regular as long as $\boldsymbol{\alpha}$ is one-to-one with $\boldsymbol{\alpha}' \neq \boldsymbol{\beta}$.

(2) Cone: Here we fix a point (say the origin) as the vertex, let $\boldsymbol{\alpha}$ be a curve with $\boldsymbol{\alpha} \times \boldsymbol{\alpha}' \neq \mathbf{0}$, and let $\boldsymbol{\beta} = -\boldsymbol{\alpha}$. Obviously, this fails to be a regular surface at the vertex (when $v = 1$), but $\mathbf{x}_u \times \mathbf{x}_v = (v-1)\boldsymbol{\alpha}(u) \times \boldsymbol{\alpha}'(u)$ is nonzero otherwise. (Note that another way to parametrize this surface would be to take $\boldsymbol{\alpha}^* = \mathbf{0}$ and $\boldsymbol{\beta}^* = \boldsymbol{\alpha}$.)

(3) Tangent developable: Let $\boldsymbol{\alpha}$ be a regular parametrized curve with nonzero curvature, and let $\boldsymbol{\beta} = \boldsymbol{\alpha}'$; that is, the rulings are the tangent lines of the curve $\boldsymbol{\alpha}$. Then $\mathbf{x}_u \times \mathbf{x}_v = -v\boldsymbol{\alpha}'(u) \times \boldsymbol{\alpha}''(u)$, so (at least locally) this is a regular parametrized surface away from the directrix. ∇

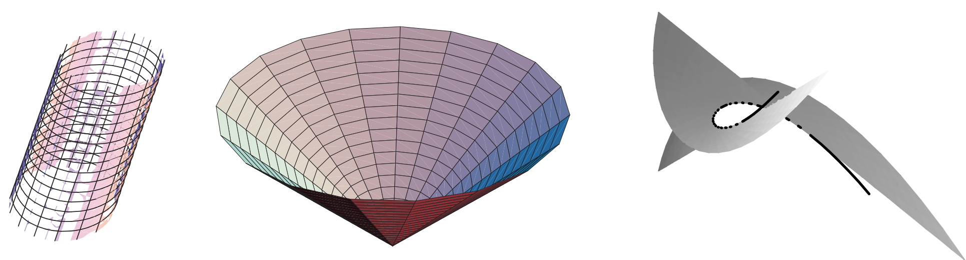

FIGURE 1.5

In calculus, we learn that, given a differentiable function f, the best linear approximation to the graph $y = f(x)$ "near" $x = a$ is given by the tangent line $y = f'(a)(x-a) + f(a)$, and similarly in higher dimensions. In the case of a regular parametrized surface, it seems reasonable that the tangent plane at $P = \mathbf{x}(u_0, v_0)$ should contain the tangent vector to the u-curve $\boldsymbol{\alpha}_1(u) = \mathbf{x}(u, v_0)$ at $u = u_0$ and the tangent vector to the v-curve $\boldsymbol{\alpha}_2(v) = \mathbf{x}(u_0, v)$ at $v = v_0$. That is, the tangent plane should contain the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$, each evaluated at (u_0, v_0). Now, since $\mathbf{x}_u \times \mathbf{x}_v \neq \mathbf{0}$ by hypothesis, the vectors $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent and must therefore span a plane. We now make this an official

Definition. Let M be a regular parametrized surface, and let $P \in M$. Then choose a regular parametrization $\mathbf{x}\colon U \to M \subset \mathbb{R}^3$ with $P = \mathbf{x}(u_0, v_0)$. We define the *tangent plane* of M at P to be the subspace $T_P M$ spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$ (evaluated at (u_0, v_0)).

Remark. The alert reader may wonder what happens if two people pick two different such local parametrizations of M near P. Do they both provide the same plane $T_P M$? This sort of question is very common in differential geometry, and is not one we intend to belabor in this introductory course. However, to get a feel for how such arguments go, the reader may work Exercise 15.

There are two unit vectors orthogonal to the tangent plane $T_P M$. Given a regular parametrization $\mathbf{x}$, we know that $\mathbf{x}_u \times \mathbf{x}_v$ is a nonzero vector orthogonal to the plane spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$; we obtain the corresponding unit vector by taking

$$\mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}.$$

This is called the *unit normal* of the parametrized surface.

Example 4. We know from basic geometry and vector calculus that the unit normal of the unit sphere centered at the origin should be the position vector itself. This is in fact what we discovered in Example 1(d). ∇

Example 5. Consider the helicoid given in Example 1(b). Then, as we saw, $\mathbf{x}_u \times \mathbf{x}_v = (b \sin v, -b \cos v, u)$, and $\mathbf{n} = \dfrac{1}{\sqrt{u^2 + b^2}}(b \sin v, -b \cos v, u)$. As we move along a ruling $v = v_0$, the normal starts horizontal at $u = 0$ (where the surface becomes vertical) and rotates in the plane orthogonal to the ruling, becoming more and more vertical as we move out the ruling. ∇

We saw in Chapter 1 that the geometry of a space curve is best understood by calculating (at least in principle) with an arclength parametrization. It would be nice, analogously, if we could find a parametrization $\mathbf{x}(u, v)$ of a surface so that $\mathbf{x}_u$ and $\mathbf{x}_v$ form an orthonormal basis at each point. We'll see later that this can happen only very rarely. But it makes it natural to introduce what is classically called the *first fundamental form*, $\mathrm{I}_P(\mathbf{U}, \mathbf{V}) = \mathbf{U} \cdot \mathbf{V}$, for $\mathbf{U}, \mathbf{V} \in T_P M$. Working in a parametrization, we have the natural basis $\{\mathbf{x}_u, \mathbf{x}_v\}$, and so we define

$$\begin{aligned}
E &= \mathrm{I}_P(\mathbf{x}_u, \mathbf{x}_u) = \mathbf{x}_u \cdot \mathbf{x}_u \\
F &= \mathrm{I}_P(\mathbf{x}_u, \mathbf{x}_v) = \mathbf{x}_u \cdot \mathbf{x}_v = \mathbf{x}_v \cdot \mathbf{x}_u = \mathrm{I}_P(\mathbf{x}_v, \mathbf{x}_u) \\
G &= \mathrm{I}_P(\mathbf{x}_v, \mathbf{x}_v) = \mathbf{x}_v \cdot \mathbf{x}_v,
\end{aligned}$$

and it is often convenient to put these in as entries of a (symmetric) matrix:

$$\mathrm{I}_P = \begin{bmatrix} E & F \\ F & G \end{bmatrix}.$$

Then, given tangent vectors $\mathbf{U} = a\mathbf{x}_u + b\mathbf{x}_v$ and $\mathbf{V} = c\mathbf{x}_u + d\mathbf{x}_v \in T_P M$, we have

$$\mathbf{U} \cdot \mathbf{V} = \mathrm{I}_P(\mathbf{U}, \mathbf{V}) = (a\mathbf{x}_u + b\mathbf{x}_v) \cdot (c\mathbf{x}_u + d\mathbf{x}_v) = E(ac) + F(ad + bc) + G(bd).$$

In particular, $\|\mathbf{U}\|^2 = \mathrm{I}_P(\mathbf{U}, \mathbf{U}) = Ea^2 + 2Fab + Gb^2$.

Suppose M and M^* are surfaces. We say they are *locally isometric* if for each $P \in M$ there are a regular parametrization $\mathbf{x}\colon U \to M$ with $\mathbf{x}(u_0, v_0) = P$ and a regular parametrization $\mathbf{x}^*\colon U \to M^*$ (using the same domain $U \subset \mathbb{R}^2$) with the property that $\mathrm{I}_P = \mathrm{I}^*_{P^*}$ whenever $P = \mathbf{x}(u, v)$ and $P^* = \mathbf{x}^*(u, v)$ for some $(u, v) \in U$. That is, the function $\mathbf{f} = \mathbf{x}^* \circ \mathbf{x}^{-1}\colon \mathbf{x}(U) \to \mathbf{x}^*(U)$ is a one-to-one correspondence that preserves the first fundamental form and is therefore distance-preserving (see Exercise 2).

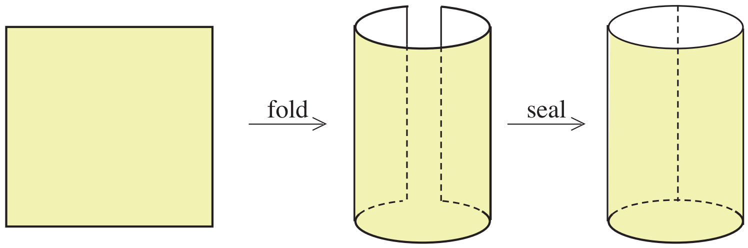

FIGURE 1.6

Example 6. Parametrize a portion of the plane (say, a piece of paper) by $\mathbf{x}(u, v) = (u, v, 0)$ and a portion of a cylinder by $\mathbf{x}^*(u, v) = (\cos u, \sin u, v)$. Then it is easy to calculate that $E = E^* = 1$, $F = F^* = 0$, and $G = G^* = 1$, so these surfaces, pictured in Figure 1.6, are locally isometric. On the other hand, if we let u vary from 0 to 2π, the rectangle and the cylinder are not *globally* isometric because points far away in the rectangle can become very close (or identical) in the cylinder. ∇

If $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$ is a curve on the parametrized surface M with $\boldsymbol{\alpha}(t_0) = \mathbf{x}(u_0, v_0) = P$, then it is an immediate consequence of the chain rule, Theorem 2.2 of the Appendix, that

$$\boldsymbol{\alpha}'(t_0) = u'(t_0)\mathbf{x}_u(u_0, v_0) + v'(t_0)\mathbf{x}_v(u_0, v_0).$$

(Customarily we will write simply $\mathbf{x}_u$, the point (u_0, v_0) at which it is evaluated being assumed.) That is, if the tangent vector $(u'(t_0), v'(t_0))$ back in the "parameter space" is (a, b), then the tangent vector to $\boldsymbol{\alpha}$ at P is the corresponding linear combination $a\mathbf{x}_u + b\mathbf{x}_v$. In fancy terms, this is merely a consequence of the linearity of the derivative of $\mathbf{x}$. We say a parametrization $\mathbf{x}(u, v)$ is *conformal* if angles measured in the

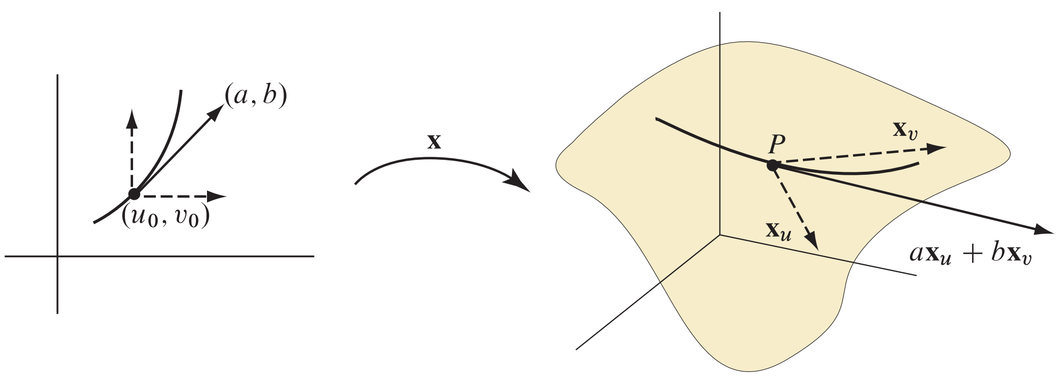

FIGURE 1.7

uv-plane agree with corresponding angles in $T_P M$ for all P. We leave it to the reader to check in Exercise 6 that this is equivalent to the conditions $E = G$, $F = 0$.

Since

$$\begin{bmatrix} E & F \\ F & G \end{bmatrix} = \begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix} = \begin{bmatrix} | & | \\ \mathbf{x}_u & \mathbf{x}_v \\ | & | \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} | & | \\ \mathbf{x}_u & \mathbf{x}_v \\ | & | \end{bmatrix},$$

we have

$$EG - F^2 = \det\left(\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v \end{bmatrix}\right) = \det\left(\begin{bmatrix} \mathbf{x}_u \cdot \mathbf{x}_u & \mathbf{x}_u \cdot \mathbf{x}_v & 0 \\ \mathbf{x}_v \cdot \mathbf{x}_u & \mathbf{x}_v \cdot \mathbf{x}_v & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}\right) = \left(\det\begin{bmatrix} | & | & | \\ \mathbf{x}_u & \mathbf{x}_v & \mathbf{n} \\ | & | & | \end{bmatrix}\right)^2,$$

which is the square of the volume of the parallelepiped spanned by $\mathbf{x}_u$, $\mathbf{x}_v$, and $\mathbf{n}$. Since $\mathbf{n}$ is a unit vector orthogonal to the plane spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$, this is, in turn, the square of the area of the parallelogram spanned by $\mathbf{x}_u$ and $\mathbf{x}_v$. That is,

$$EG - F^2 = \|\mathbf{x}_u \times \mathbf{x}_v\|^2 > 0.$$

We remind the reader that we obtain the *surface area* of the parametrized surface $\mathbf{x}\colon U \to M$ by calculating the double integral

$$\int_U \|\mathbf{x}_u \times \mathbf{x}_v\| du dv = \int_U \sqrt{EG - F^2} du dv.$$

EXERCISES 2.1

1. Derive the formula given in Example 1(e) for the parametrization of the unit sphere.

♯2. Suppose $\boldsymbol{\alpha}(t) = \mathbf{x}(u(t), v(t))$, $a \le t \le b$, is a parametrized curve on a surface M. Show that

$$\begin{aligned} \text{length}(\boldsymbol{\alpha}) &= \int_a^b \sqrt{\mathrm{I}_{\boldsymbol{\alpha}(t)}\big(\boldsymbol{\alpha}'(t), \boldsymbol{\alpha}'(t)\big)} dt \\ &= \int_a^b \sqrt{E(u(t), v(t))(u'(t))^2 + 2F(u(t), v(t))u'(t)v'(t) + G(u(t), v(t))(v'(t))^2} dt. \end{aligned}$$

Conclude that if $\boldsymbol{\alpha} \subset M$ and $\boldsymbol{\alpha}^* \subset M^*$ are corresponding paths in locally isometric surfaces, then $\text{length}(\boldsymbol{\alpha}) = \text{length}(\boldsymbol{\alpha}^*)$.

3. Compute I (i.e., E, F, and G) for the following parametrized surfaces.
   *a. the sphere of radius a: $\mathbf{x}(u, v) = a(\sin u \cos v, \sin u \sin v, \cos u)$
   b. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a)$
   c. the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$
   *d. the catenoid: $\mathbf{x}(u, v) = a(\cosh u \cos v, \cosh u \sin v, u)$

4. Find the surface area of the following parametrized surfaces.
   *a. the torus: $\mathbf{x}(u, v) = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)$ $(0 < b < a)$, $0 \le u, v \le 2\pi$
   b. a portion of the helicoid: $\mathbf{x}(u, v) = (u\cos v, u\sin v, bv)$, $1 < u < 3$, $0 \le v \le 2\pi$
   c. a zone of a sphere[3]: $\mathbf{x}(u, v) = a(\sin u \cos v, \sin u \sin v, \cos u)$, $0 \le u_0 \le u \le u_1 \le \pi$, $0 \le v \le 2\pi$

[3] You should obtain the remarkable result that the surface area of the portion of a sphere between two parallel planes depends only on the distance between the planes, not on where you locate them.

*5. Show that if all the normal lines to a surface pass through a fixed point, then the surface is (a portion of) a sphere. (By the normal line to M at P we mean the line passing through P with direction vector the unit normal at P.)

6. Check that the parametrization $\mathbf{x}(u,v)$ is conformal if and only if $E = G$ and $F = 0$. (Hint: For $\Longrightarrow$, choose *two* convenient pairs of orthogonal directions.)

*7. Check that a parametrization preserves area and is conformal if and only if it is a local isometry.

*8. Check that the parametrization of the unit sphere by stereographic projection (see Example 1(e)) is conformal.

9. *(Lambert's cylindrical projection)* Project the unit sphere (except for the north and south poles) radially outward to the cylinder of radius 1 by sending (x,y,z) to $(x/\sqrt{x^2+y^2}, y/\sqrt{x^2+y^2}, z)$. Check that this map preserves area locally, but is neither a local isometry nor conformal. (Hint: Let $\mathbf{x}(u,v)$ be the spherical coordinates parametrization of the sphere, and consider $\mathbf{x}^*(u,v) = (\cos v, \sin v, \cos u)$. Compare the parallelogram formed by $\mathbf{x}_u$ and $\mathbf{x}_v$ with the parallelogram formed by $\mathbf{x}^*_u$ and $\mathbf{x}^*_v$.)

♯10. Consider the "pacman" region M given by $\mathbf{x}(u,v) = (u\cos v, u\sin v, 0), 0 \le u \le R, 0 \le v \le V$, with $V < 2\pi$. Let $c = V/2\pi$. Let M^* be given by the parametrization

$$\mathbf{x}^*(u,v) = \big(cu\cos(v/c), cu\sin(v/c), \sqrt{1-c^2}u\big), \quad 0 \le u \le R,\ 0 \le v \le V.$$

Compute that $E = E^*$, $F = F^*$, and $G = G^*$, and conclude that the mapping $\mathbf{f} = \mathbf{x}^* \circ \mathbf{x}^{-1} \colon M \to M^*$ is a local isometry. Describe this mapping in concrete geometric terms.

11. Consider the hyperboloid of one sheet, M, given by the equation $x^2 + y^2 - z^2 = 1$.
   a. Show that $\mathbf{x}(u,v) = (\cosh u\cos v, \cosh u\sin v, \sinh u)$, $u \in \mathbb{R}$, $0 \le v < 2\pi$, gives a parametrization of M as a surface of revolution.
   *b. Find two parametrizations of M as a ruled surface $\boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u)$.
   c. Show that $\mathbf{x}(u,v) = \left(\dfrac{uv+1}{uv-1}, \dfrac{u-v}{uv-1}, \dfrac{u+v}{uv-1}\right)$ gives a parametrization of M where *both* sets of parameter curves are rulings.

♯12. Given a ruled surface M parametrized by $\mathbf{x}(u,v) = \boldsymbol{\alpha}(u) + v\boldsymbol{\beta}(u)$ with $\boldsymbol{\alpha}' \ne 0$ and $\|\boldsymbol{\beta}\| = 1$.
   a. Check that we may assume that $\boldsymbol{\alpha}'(u) \cdot \boldsymbol{\beta}(u) = 0$ for all u. (Hint: Replace $\boldsymbol{\alpha}(u)$ with $\boldsymbol{\alpha}(u) + t(u)\boldsymbol{\beta}(u)$ for a suitable function t.)
   b. Suppose, moreover, that $\boldsymbol{\alpha}'(u)$, $\boldsymbol{\beta}(u)$, and $\boldsymbol{\beta}'(u)$ are linearly dependent for every u. Conclude that $\boldsymbol{\beta}'(u) = \lambda(u)\boldsymbol{\alpha}'(u)$ for some function λ. Prove that:
      (i) If $\lambda(u) = 0$ for all u, then M is a cylinder.
      (ii) If λ is a nonzero constant, then M is a cone.
      (iii) If λ and λ' are both nowhere zero, then M is a tangent developable. (Hint: Find the directrix.)

13. *(The Mercator projection)* Mercator developed his system for mapping the earth, as pictured in Figure 1.8, in 1569, about a century before the advent of calculus. We want a parametrization $\mathbf{x}(u,v)$ of the sphere, $u \in \mathbb{R}$, $v \in (-\pi,\pi)$, so that the u-curves are the longitudes and so that the parametrization is conformal. Letting (ϕ,θ) be the usual spherical coordinates, write $\phi = f(u)$ and $\theta = v$. Show that

FIGURE 1.8

conformality and symmetry about the equator will dictate $f(u) = 2\arctan(e^{-u})$. Deduce that

$$\mathbf{x}(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u).$$

(Cf. Example 2 in Section 1 of Chapter 1.)

14. A parametrization $\mathbf{x}(u, v)$ is called a *Tschebyschev net* if the opposite sides of any quadrilateral formed by the coordinate curves have equal length.
   a. Prove that this occurs if and only if $\dfrac{\partial E}{\partial v} = \dfrac{\partial G}{\partial u} = 0$. (Hint: Express the length of the u-curves, $u_0 \le u \le u_1$, as an integral and use the fact that this length is independent of v.)
   b. Prove that we can locally reparametrize by $\tilde{\mathbf{x}}(\tilde{u}, \tilde{v})$ so as to obtain $\tilde{E} = \tilde{G} = 1$, $\tilde{F} = \cos\theta(\tilde{u}, \tilde{v})$ (so that the $\tilde{u}$- and $\tilde{v}$-curves are parametrized by arclength and meet at angle θ). (Hint: Choose $\tilde{u}$ as a function of u so that $\tilde{\mathbf{x}}_{\tilde{u}} = \mathbf{x}_u \big/ (d\tilde{u}/du)$ has unit length.)

15. Suppose $\mathbf{x}$ and $\mathbf{y}$ are two parametrizations of a surface M near P. Say $\mathbf{x}(u_0, v_0) = P = \mathbf{y}(s_0, t_0)$. Prove that $\operatorname{Span}(\mathbf{x}_u, \mathbf{x}_v) = \operatorname{Span}(\mathbf{y}_s, \mathbf{y}_t)$ (where the partial derivatives are all evaluated at the obvious points). (Hint: $\mathbf{f} = \mathbf{x}^{-1} \circ \mathbf{y}$ gives a $\mathcal{C}^1$ map from an open set around (s_0, t_0) to an open set around (u_0, v_0). Apply the chain rule to show $\mathbf{y}_s, \mathbf{y}_t \in \operatorname{Span}(\mathbf{x}_u, \mathbf{x}_v)$.)

16. (A programmable calculator, Maple, or Mathematica will be needed for parts of this problem.) A catenoid, as pictured in Figure 1.9, is parametrized by

$$\mathbf{x}(u, v) = (a\cosh u \cos v, a\cosh u \sin v, au), \quad u \in \mathbb{R},\ 0 \le v < 2\pi \quad (a > 0 \text{ fixed}).$$

   *a. Compute the surface area of that portion of the catenoid given by $|u| \le 1/a$. (Hint: $\cosh^2 u = \frac{1}{2}(1 + \cosh 2u)$.)
   b. Find the number $R_0 > 0$ so that for every $R \ge R_0$, there is at least one catenoid whose boundary is the pair of parallel circles $x^2 + y^2 = R^2$, $|z| = 1$. (Hint: Graph $f(t) = t\cosh(1/t)$.)
   c. For $R \ge R_0$, compare the area of the catenoid(s) with $2\pi R^2$ (the area of the pair of disks filling in the circles). For what values of R does the pair of disks have the least area? (You should display the results of your investigation in either a graph or a table.)

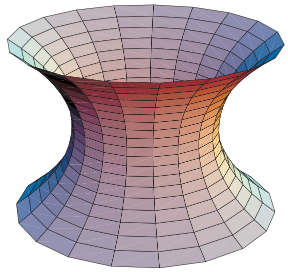

FIGURE 1.9

d. (For extra credit) Show that as $R \to \infty$, the area of the inner catenoid is asymptotic to $2\pi R^2$ and the area of the outer catenoid is asymptotic to $4\pi R$.

17. There are two obvious families of circles on a torus. Find a third family. (Hint: Look for a plane that is tangent to the torus at *two* points. Using the parametrization of the torus, you should be able to find equations (either parametric or cartesian) for the curve in which the bitangent plane intersects the torus.)

2. The Gauss Map and the Second Fundamental Form

Given a regular parametrized surface M, the function $\mathbf{n}: M \to \Sigma$ that assigns to each point $P \in M$ the unit normal $\mathbf{n}(P)$, as pictured in Figure 2.1, is called the *Gauss map* of M. As we shall see in this chapter,

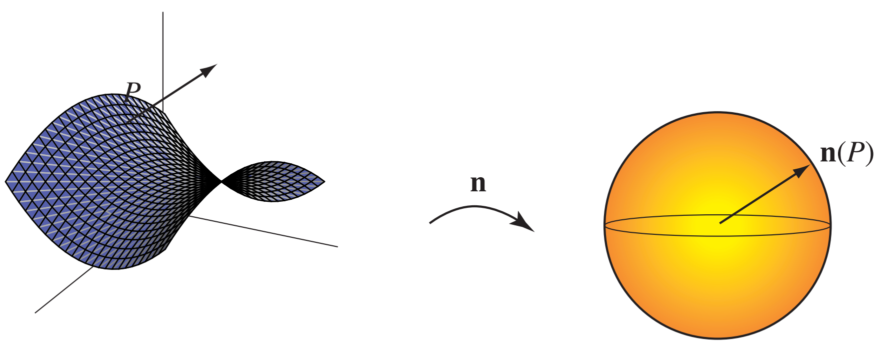

FIGURE 2.1

most of the geometric information about our surface M is encapsulated in the mapping $\mathbf{n}$.

Example 1. A few basic examples are these.

(a) On a plane, the tangent plane never changes, so the Gauss map is a constant.
(b) On a cylinder, the tangent plane is constant along the rulings, so the Gauss map sends the entire surface to an equator of the sphere.
(c) On a sphere centered at the origin, the Gauss map is merely the (normalized) position vector.
(d) On a saddle surface (as pictured in Figure 2.1), the Gauss map appears to "reverse orientation": As we move counterclockwise in a small circle around P, we see that the unit vector $\mathbf{n}$ turns clockwise around $\mathbf{n}(P)$. ∇