a differential equation that θ must satisfy, involving only κ. Note that the path of the rear wheel will obviously depend on the initial condition $\theta(0)$. In all but the simplest of cases, it may be impossible to solve the differential equation explicitly.)

3. Some Global Results

3.1. Space Curves. The fundamental notion in geometry (see Section 1 of the Appendix) is that of congruence: When do two figures differ merely by a rigid motion? If the curve $\boldsymbol{\alpha}^*$ is obtained from the curve $\boldsymbol{\alpha}$ by performing a rigid motion (composition of a translation and a rotation), then the Frenet frames at corresponding points differ by that same rigid motion, and the twisting of the frames (which is what gives curvature and torsion) should be the same. (Note that a reflection will not affect the curvature, but will change the sign of the torsion.)

Theorem 3.1 (Fundamental Theorem of Curve Theory)**.** *Two space curves C and C^* with nonzero curvature are congruent (i.e., differ by a rigid motion) if and only if the corresponding arclength parametrizations $\boldsymbol{\alpha}, \boldsymbol{\alpha}^*\colon [0, L] \to \mathbb{R}^3$ have the property that $\kappa(s) = \kappa^*(s)$ and $\tau(s) = \tau^*(s)$ for all $s \in [0, L]$.*

Proof. Suppose $\boldsymbol{\alpha}^* = \Psi \circ \boldsymbol{\alpha}$ for some rigid motion $\Psi\colon \mathbb{R}^3 \to \mathbb{R}^3$, so $\Psi(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ for some $\mathbf{b} \in \mathbb{R}^3$ and some 3×3 orthogonal matrix A with $\det A > 0$. Then $\boldsymbol{\alpha}^*(s) = A\boldsymbol{\alpha}(s) + \mathbf{b}$, so $\|\boldsymbol{\alpha}^{*\prime}(s)\| = \|A\boldsymbol{\alpha}'(s)\| = 1$, since A is orthogonal. Therefore, $\boldsymbol{\alpha}^*$ is likewise arclength-parametrized, and $\mathbf{T}^*(s) = A\mathbf{T}(s)$. Differentiating again, $\kappa^*(s)\mathbf{N}^*(s) = \kappa(s)A\mathbf{N}(s)$. Since A is orthogonal, $A\mathbf{N}(s)$ is a unit vector, and so $\mathbf{N}^*(s) = A\mathbf{N}(s)$ and $\kappa^*(s) = \kappa(s)$. But then $\mathbf{B}^*(s) = \mathbf{T}^*(s) \times \mathbf{N}^*(s) = A\mathbf{T}(s) \times A\mathbf{N}(s) = A(\mathbf{T}(s) \times \mathbf{N}(s)) = A\mathbf{B}(s)$, inasmuch as orthogonal matrices map orthonormal bases to orthonormal bases and $\det A > 0$ insures that orientation is preserved as well (i.e., right-handed bases map to right-handed bases). Last, $\mathbf{B}^{*\prime}(s) = -\tau^*(s)\mathbf{N}^*(s)$ and $\mathbf{B}^{*\prime}(s) = A\mathbf{B}'(s) = -\tau(s)A\mathbf{N}(s) = -\tau(s)\mathbf{N}^*(s)$, so $\tau^*(s) = \tau(s)$, as required.

Conversely, suppose $\kappa = \kappa^*$ and $\tau = \tau^*$. We now define a rigid motion Ψ as follows. Let A be the unique orthogonal matrix so that $A\mathbf{T}(0) = \mathbf{T}^*(0)$, $A\mathbf{N}(0) = \mathbf{N}^*(0)$, and $A\mathbf{B}(0) = \mathbf{B}^*(0)$, and let $\mathbf{b} = \boldsymbol{\alpha}^*(0) - A\boldsymbol{\alpha}(0)$. A also has positive determinant, since both orthonormal bases are right-handed. Set $\tilde{\boldsymbol{\alpha}} = \Psi \circ \boldsymbol{\alpha}$. We now claim that $\boldsymbol{\alpha}^*(s) = \tilde{\boldsymbol{\alpha}}(s)$ for all $s \in [0, L]$. Note, by our argument in the first part of the proof, that $\tilde{\kappa} = \kappa = \kappa^*$ and $\tilde{\tau} = \tau = \tau^*$. Consider

$$f(s) = \tilde{\mathbf{T}}(s) \cdot \mathbf{T}^*(s) + \tilde{\mathbf{N}}(s) \cdot \mathbf{N}^*(s) + \tilde{\mathbf{B}}(s) \cdot \mathbf{B}^*(s).$$

We now differentiate f, using the Frenet formulas.

$$\begin{aligned} f'(s) &= \big(\tilde{\mathbf{T}}'(s) \cdot \mathbf{T}^*(s) + \tilde{\mathbf{T}}(s) \cdot \mathbf{T}^{*\prime}(s)\big) + \big(\tilde{\mathbf{N}}'(s) \cdot \mathbf{N}^*(s) + \tilde{\mathbf{N}}(s) \cdot \mathbf{N}^{*\prime}(s)\big) \\ &\qquad\qquad + \big(\tilde{\mathbf{B}}'(s) \cdot \mathbf{B}^*(s) + \tilde{\mathbf{B}}(s) \cdot \mathbf{B}^{*\prime}(s)\big) \\ &= \kappa(s)\big(\tilde{\mathbf{N}}(s) \cdot \mathbf{T}^*(s) + \tilde{\mathbf{T}}(s) \cdot \mathbf{N}^*(s)\big) - \kappa(s)\big(\tilde{\mathbf{T}}(s) \cdot \mathbf{N}^*(s) + \tilde{\mathbf{N}}(s) \cdot \mathbf{T}^*(s)\big) \\ &\qquad + \tau(s)\big(\tilde{\mathbf{B}}(s) \cdot \mathbf{N}^*(s) + \tilde{\mathbf{N}}(s) \cdot \mathbf{B}^*(s)\big) - \tau(s)\big(\tilde{\mathbf{N}}(s) \cdot \mathbf{B}^*(s) + \tilde{\mathbf{B}}(s) \cdot \mathbf{N}^*(s)\big) \\ &= 0, \end{aligned}$$

since the first two terms cancel and the last two terms cancel. By construction, $f(0) = 3$, so $f(s) = 3$ for all $s \in [0, L]$. Since each of the individual dot products can be at most 1, the only way the sum can be 3 for all s is for each to be 1 for all s, and this in turn can happen only when $\tilde{\mathbf{T}}(s) = \mathbf{T}^*(s)$, $\tilde{\mathbf{N}}(s) = \mathbf{N}^*(s)$, and $\tilde{\mathbf{B}}(s) = \mathbf{B}^*(s)$ for all $s \in [0, L]$. In particular, since $\tilde{\boldsymbol{\alpha}}'(s) = \tilde{\mathbf{T}}(s) = \mathbf{T}^*(s) = \boldsymbol{\alpha}^{*\prime}(s)$ and $\tilde{\boldsymbol{\alpha}}(0) = \boldsymbol{\alpha}^*(0)$, it follows that $\tilde{\boldsymbol{\alpha}}(s) = \boldsymbol{\alpha}^*(s)$ for all $s \in [0, L]$, as we wished to show. $\square$

Remark. The latter half of this proof can be replaced by asserting the uniqueness of solutions of a system of differential equations, as we will see in a moment. Also see Exercise A.3.1 for a matrix-computational version of the proof we just did.

Example 1. We now see that the only curves with constant κ and τ are circular helices. ∇

Perhaps more interesting is the existence question: Given continuous functions $\kappa, \tau\colon [0, L] \to \mathbb{R}$ (with κ everywhere positive), is there a space curve with those as its curvature and torsion? The answer is yes, and this is an immediate consequence of the fundamental existence theorem for differential equations, Theorem 3.1 of the Appendix. That is, we let

$$F(s) = \begin{bmatrix} | & | & | \\ \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}(s) \\ | & | & | \end{bmatrix} \qquad \text{and} \qquad K(s) = \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}.$$

Then integrating the linear system of ordinary differential equations $F'(s) = F(s)K(s)$, $F(0) = F_0$, gives us the Frenet frame everywhere along the curve, and we recover $\boldsymbol{\alpha}$ by integrating $\mathbf{T}(s)$.

We turn now to the concept of *total curvature* of a *closed* space curve, which is the integral of its curvature. That is, if $\boldsymbol{\alpha}\colon [0, L] \to \mathbb{R}^3$ is an arclength-parametrized curve with $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(L)$, $\boldsymbol{\alpha}'(0) = \boldsymbol{\alpha}'(L)$, and $\boldsymbol{\alpha}''(0) = \boldsymbol{\alpha}''(L)$, then its total curvature is $\displaystyle\int_0^L \kappa(s)\,ds$. This quantity can be interpreted geometrically as follows: The *Gauss map* of $\boldsymbol{\alpha}$ is the map to the unit sphere, Σ, given by the unit tangent vector $\mathbf{T}\colon [0, L] \to \Sigma$; its image, Γ, is classically called the *tangent indicatrix* of $\boldsymbol{\alpha}$. Observe that—provided the Gauss map is one-

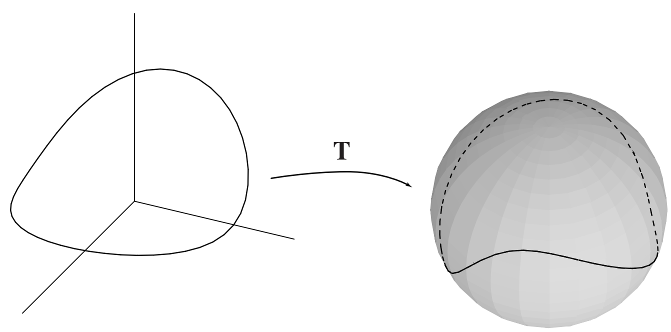

FIGURE 3.1

to-one—the length of Γ is the total curvature of $\boldsymbol{\alpha}$, since $\text{length}(\Gamma) = \displaystyle\int_0^L \|\mathbf{T}'(s)\|\,ds = \int_0^L \kappa(s)\,ds$. More generally, this integral is the length of Γ "counting multiplicities."

A preliminary question to ask is this: What curves Γ in the unit sphere can be the Gauss map of some *closed* space curve $\boldsymbol{\alpha}$? Since $\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}(0) + \displaystyle\int_0^s \mathbf{T}(u)\,du$, we see that a necessary and sufficient condition is that $\displaystyle\int_0^L \mathbf{T}(s)\,ds = \mathbf{0}$. (Note, however, that this depends on the arclength parametrization of the original curve and is not a parametrization-independent condition on the image curve $\Gamma \subset \Sigma$.) We do, nevertheless, have the following geometric consequence of this condition. For any (unit) vector $\mathbf{A}$, we have

$$0 = \mathbf{A} \cdot \int_0^L \mathbf{T}(s)ds = \int_0^L (\mathbf{T}(s) \cdot \mathbf{A})ds,$$

and so the average value of $\mathbf{T} \cdot \mathbf{A}$ must be 0. In particular, the tangent indicatrix must cross the great circle with normal vector $\mathbf{A}$. That is, if the curve Γ is to be a tangent indicatrix, it must be "balanced" with respect to every direction $\mathbf{A}$. It is natural to ask for the shortest curve(s) with this property.

If $\boldsymbol{\xi} \in \Sigma$, let $\boldsymbol{\xi}^\perp$ denote the oriented great circle with normal vector $\boldsymbol{\xi}$. (By this we mean that we go around the circle $\boldsymbol{\xi}^\perp$ so that at $\mathbf{x}$, the tangent vector $\mathbf{T}$ points so that $\mathbf{x}, \mathbf{T}, \boldsymbol{\xi}$ form a right-handed basis for $\mathbb{R}^3$.)

Proposition 3.2 (Crofton's formula)**.** *Let Γ be a piecewise-$\mathcal{C}^1$ curve on the sphere. Then*

$$\begin{aligned} \text{length}(\Gamma) &= \frac{1}{4}\int_\Sigma \#(\Gamma \cap \boldsymbol{\xi}^\perp)d\boldsymbol{\xi} \\ &= \pi \times (\textit{the average number of intersections of } \Gamma \textit{ with all great circles}). \end{aligned}$$

(Here $d\boldsymbol{\xi}$ represents the usual element of surface area on Σ.)

Proof. We leave this to the reader in Exercise 11. □

Remark. Although we don't stop to justify it here, the set of $\boldsymbol{\xi}$ for which $\#(\Gamma \cap \boldsymbol{\xi}^\perp)$ is infinite is a set of measure zero, and so the integral makes sense.

Applying this to the case of the tangent indicatrix of a closed space curve, we deduce the following classical result.

Theorem 3.3 (Fenchel)**.** *The total curvature of any closed space curve is at least 2π, and equality holds if and only if the curve is a (convex) planar curve.*

Proof. Let Γ be the tangent indicatrix of our space curve. If C is a closed plane curve, then Γ is a great circle on the sphere. As we shall see in the next section, convexity of the curve can be interpreted as saying $\kappa > 0$ everywhere, so the tangent indicatrix traverses the great circle exactly once and $\int_C \kappa ds = 2\pi$ (cf. Theorem 3.5 in the next section).

To prove the converse, note that, by our earlier remarks, Γ must cross $\boldsymbol{\xi}^\perp$ for almost every $\boldsymbol{\xi} \in \Sigma$ and hence must intersect it at least twice, and so it follows from Proposition 3.2 that $\int_C \kappa ds = \text{length}(\Gamma) \geq \frac{1}{4}(2)(4\pi) = 2\pi$. Now, we claim that if Γ is a connected, closed curve in Σ of length $\leq 2\pi$, then Γ lies in a closed hemisphere. It will follow, then, that if Γ is a tangent indicatrix of length 2π, it must be a great circle. (For if Γ lies in the hemisphere $\mathbf{A} \cdot \mathbf{x} \geq 0$, $\int_0^L \mathbf{T}(s) \cdot \mathbf{A} ds = 0$ forces $\mathbf{T} \cdot \mathbf{A} = 0$, so Γ is the great circle $\mathbf{A} \cdot \mathbf{x} = 0$.) It follows that the curve is planar and the tangent indicatrix traverses the great circle precisely one time, which means that $\kappa > 0$ and the curve is convex. (See the next section for more details on this.)

To prove the claim, we proceed as follows. Suppose length$(\Gamma) \leq 2\pi$. Choose P and Q in Γ so that the arcs $\Gamma_1 = \overset{\frown}{PQ}$ and $\Gamma_2 = \overset{\frown}{QP}$ have the same length. Choose N bisecting the shorter great circle arc from P to Q, as shown in Figure 3.2. For convenience, we rotate the picture so that N is the north pole of the sphere. Suppose now that the curve Γ_1 were to enter the southern hemisphere; let $\overline{\Gamma}_1$ denote the reflection of Γ_1

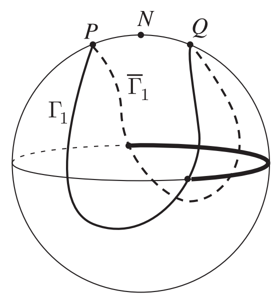

FIGURE 3.2

across the north pole (following arcs of great circle through N). Now, $\Gamma_1 \cup \overline{\Gamma}_1$ is a closed curve containing a pair of antipodal points and therefore is longer than a great circle. (See Exercise 1.) Since $\Gamma_1 \cup \overline{\Gamma}_1$ has the same length as Γ, we see that length$(\Gamma) > 2\pi$, which is a contradiction. Therefore Γ indeed lies in the northern hemisphere. $\square$

We now sketch the proof of a result that has led to many interesting questions in higher dimensions. We say a simple (non-self-intersecting) closed[4] space curve is *knotted* if we cannot fill it in with a disk.

Theorem 3.4 (Fáry-Milnor). *If a simple closed space curve is knotted, then its total curvature is at least 4π.*

Sketch of proof. Suppose the total curvature of C is less than 4π. Then the average number $\#(\Gamma \cap \boldsymbol{\xi}^{\perp}) < 4$. Since this is generically an even number ≥ 2 (whenever the great circle isn't tangent to Γ), there must be an open set of $\boldsymbol{\xi}$'s for which we have $\#(\Gamma \cap \boldsymbol{\xi}^{\perp}) = 2$. Choose one such, $\boldsymbol{\xi}_0$. This means that the tangent vector to C is only perpendicular to $\boldsymbol{\xi}_0$ twice, so the function $f(\mathbf{x}) = \mathbf{x} \cdot \boldsymbol{\xi}_0$ has only two critical points. That is, the planes perpendicular to $\boldsymbol{\xi}_0$ will intersect C either in a single point (at the maximum and minimum points of f) or in exactly two points (by Rolle's Theorem). Now, by moving these planes from the bottom of C to the top, joining the two intersection points in each plane with a line segment, we fill in a disk, so C is unknotted. $\square$

3.2. Plane Curves. We conclude this chapter with some results on plane curves. Now we assign a sign to the curvature: Given an arclength-parametrized curve $\boldsymbol{\alpha}$, (re)define $\mathbf{N}(s)$ so that $\{\mathbf{T}(s), \mathbf{N}(s)\}$ is a right-handed basis for $\mathbb{R}^2$ (i.e., one turns counterclockwise from $\mathbf{T}(s)$ to $\mathbf{N}(s)$), and then set $\kappa(s) = \mathbf{T}'(s) \cdot \mathbf{N}(s)$, from which it follows that $\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s)$ (why?), as before. So $\kappa > 0$ when $\mathbf{T}$ is twisting counterclockwise and $\kappa < 0$ when $\mathbf{T}$ is twisting clockwise. Although the total curvature $\int_C |\kappa(s)|ds$ of a simple closed plane curve may be quite a bit larger than 2π, it is intuitively plausible that the tangent vector must make precisely one full rotation, either counterclockwise or clockwise, and thus we have

Theorem 3.5 (Hopf Umlaufsatz). *If C is a simple closed plane curve, then $\int_C \kappa\, ds = \pm 2\pi$, the $+$ occurring when C is oriented counterclockwise and $-$ when it's oriented clockwise.*

The crucial ingredient is to keep track of a *continuous* total angle through which the tangent vector has turned. That is, we need the following

[4]To be more careful here, if $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^3$ is a parametrization with $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$, then $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(u)$ occurs *only* when $\{t,u\} = \{a,b\}$.

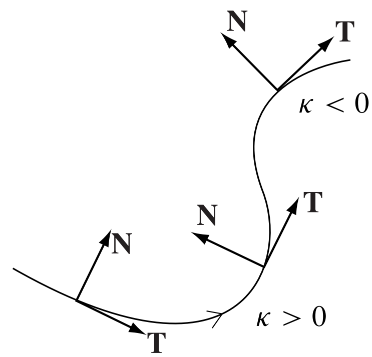

FIGURE 3.3

Lemma 3.6. *Let* $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^2$ *be a* $\mathcal{C}^1$*, regular parametrized plane curve. Then there is a* $\mathcal{C}^1$ *function* $\theta\colon [a,b] \to \mathbb{R}$ *so that* $\mathbf{T}(t) = \big(\cos\theta(t), \sin\theta(t)\big)$ *for all* $t \in [a,b]$*. Moreover, for any two such functions,* θ *and* θ^**, we have* $\theta(b) - \theta(a) = \theta^*(b) - \theta^*(a)$*. The number* $(\theta(b) - \theta(a))/2\pi$ *is called the* rotation index *of* $\boldsymbol{\alpha}$.

Proof. Consider the four open semicircles $U_1 = \{(x,y) \in S^1 : x > 0\}$, $U_2 = \{(x,y) \in S^1 : x < 0\}$, $U_3 = \{(x,y) \in S^1 : y > 0\}$, and $U_4 = \{(x,y) \in S^1 : y < 0\}$. Then the functions

$$\begin{aligned}
\psi_{1,n}(x,y) &= \arctan(y/x) + 2n\pi \\
\psi_{2,n}(x,y) &= \arctan(y/x) + (2n+1)\pi \\
\psi_{3,n}(x,y) &= -\arctan(x/y) + (2n+\tfrac{1}{2})\pi \\
\psi_{4,n}(x,y) &= -\arctan(x/y) + (2n-\tfrac{1}{2})\pi
\end{aligned}$$

are smooth maps $\psi_{i,n}\colon U_i \to \mathbb{R}$ with the property that $\big(\cos(\psi_{i,n}(x,y)), \sin(\psi_{i,n}(x,y))\big) = (x,y)$ for every $i = 1,2,3,4$ and $n \in \mathbb{Z}$.

Define $\theta(a)$ so that $\mathbf{T}(a) = \big(\cos\theta(a), \sin\theta(a)\big)$. Let $S = \{t \in [a,b] : \theta$ is defined and $\mathcal{C}^1$ on $[a,t]\}$, and let $t_0 = \sup S$. Suppose first that $t_0 < b$. Choose i so that $\mathbf{T}(t_0) \in U_i$, and choose $n \in \mathbb{Z}$ so that $\psi_{i,n}(\mathbf{T}(t_0)) = \lim_{t \to t_0^-} \theta(t)$. Because $\mathbf{T}$ is continuous at t_0, there is $\delta > 0$ so that $\mathbf{T}(t) \in U_i$ for all t with $|t - t_0| < \delta$. Then setting $\theta(t) = \psi_{i,n}(\mathbf{T}(t))$ for all $t_0 \le t < t_0 + \delta$ gives us a $\mathcal{C}^1$ function θ defined on $[0, t_0 + \delta/2]$, so we cannot have $t_0 < b$. (Note that $\theta(t) = \psi_{i,n}(\mathbf{T}(t))$ for all $t_0 - \delta < t < t_0$. Why?) But the same argument shows that when $t_0 = b$, the function θ is $\mathcal{C}^1$ on all of $[a,b]$.

Now, since $\mathbf{T}(b) = \mathbf{T}(a)$, we know that $\theta(b) - \theta(a)$ must be an integral multiple of 2π. Moreover, for any other function θ^* with the same properties, we have $\theta^*(t) = \theta(t) + 2\pi n(t)$ for some integer $n(t)$. Since θ and θ^* are both continuous, n must be a continuous function as well; since it takes on only integer values, it must be a constant function. Therefore, $\theta^*(b) - \theta^*(a) = \theta(b) - \theta(a)$, as required. □

Sketch of proof of Theorem 3.5. Note first that if $\mathbf{T}(s) = \big(\cos\theta(s), \sin\theta(s)\big)$, then $\mathbf{T}'(s) = \theta'(s)\big(-\sin\theta(s), \cos\theta(s)\big)$, so $\kappa(s) = \theta'(s)$ and $\int_0^L \kappa(s)ds = \int_0^L \theta'(s)ds = \theta(L) - \theta(0)$ is 2π times the rotation index of the closed curve $\boldsymbol{\alpha}$.

Let $\Delta = \{(s,t) : 0 \le s \le t \le L\}$. Consider the secant map $\mathbf{h}\colon \Delta \to S^1$ defined by

$$\mathbf{h}(s,t) = \begin{cases} \mathbf{T}(s), & s = t \\ -\mathbf{T}(0), & (s,t) = (0,L) \\ \dfrac{\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)}{\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\|}, & \text{otherwise} \end{cases}.$$

Then it follows from Proposition 2.6 (using Taylor's Theorem to calculate $\boldsymbol{\alpha}(t) = \boldsymbol{\alpha}(s) + (t-s)\boldsymbol{\alpha}'(s) + \dots$) that $\mathbf{h}$ is continuous. A more sophisticated version of the proof of Lemma 3.6 will establish (see Exercise 13) that there is a continuous function $\tilde{\theta}\colon \Delta \to \mathbb{R}$ so that $\mathbf{h}(s,t) = \big(\cos\tilde{\theta}(s,t), \sin\tilde{\theta}(s,t)\big)$ for all $(s,t) \in \Delta$. It then follows from Lemma 3.6 that

$$\int_C \kappa\, ds = \theta(L) - \theta(0) = \tilde{\theta}(L,L) - \tilde{\theta}(0,0) = \underbrace{\tilde{\theta}(0,L) - \tilde{\theta}(0,0)}_{N_1} + \underbrace{\tilde{\theta}(L,L) - \tilde{\theta}(0,L)}_{N_2}.$$

Rotating the curve as required, we assume that $\boldsymbol{\alpha}(0)$ is the lowest point on the curve (i.e., the one whose y-coordinate is smallest) and, then, that $\boldsymbol{\alpha}(0)$ is the origin and $\mathbf{T}(0) = \mathbf{e}_1$, as shown in Figure 3.4. (The

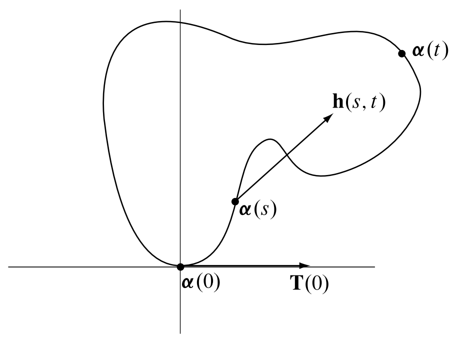

FIGURE 3.4

last may require reversing the orientation of the curve.) Now, N_1 is the angle through which the position vector of the curve turns, starting at 0 and ending at π; since the curve lies in the upper half-plane, we must have $N_1 = \pi$. But N_2 is likewise the angle through which the negative of the position vector turns, so $N_2 = N_1 = \pi$. With these assumptions, we see that the rotation index of the curve is 1. Allowing for the possible change in orientation, the rotation index must therefore be ± 1, as required. $\square$

Corollary 3.7. *If C is any closed curve with nonzero rotation index (e.g., a simple closed curve), for any point $P \in C$ there is a point $Q \in C$ where the unit tangent vector is opposite that at P.*

Proof. Let $\mathbf{T}(s) = \big(\cos\theta(s), \sin\theta(s)\big)$ for a $\mathcal{C}^1$ function $\theta\colon [0,L] \to \mathbb{R}$, as in Lemma 3.6. Say $P = \boldsymbol{\alpha}(s_0)$, and let $\theta(s_0) = \theta_0$. Since $\theta(L) - \theta(0)$ is an integer multiple of 2π, there must be $s_1 \in [0,L]$ with either $\theta(s_1) = \theta_0 + \pi$ or $\theta(s_1) = \theta_0 - \pi$. Take $Q = \boldsymbol{\alpha}(s_1)$. $\square$

Recall that one of the ways of characterizing a convex function $f\colon \mathbb{R} \to \mathbb{R}$ is that its graph lie on one side of each of its tangent lines. So we make the following

Definition. The regular closed plane curve $\boldsymbol{\alpha}$ is *convex* if it lies on one side of its tangent line at each point.

Proposition 3.8. *A simple closed regular plane curve C is convex if and only if we can choose the orientation of the curve so that $\kappa \geq 0$ everywhere.*

Remark. We leave it to the reader in Exercise 2 to give a non-simple closed curve for which this result is false.

Proof. Assume, without loss of generality, that $\mathbf{T}(0) = (1, 0)$ and the curve is oriented counterclockwise. Using the function θ constructed in Lemma 3.6, the condition that $\kappa \geq 0$ is equivalent to the condition that θ is a nondecreasing function with $\theta(L) = 2\pi$.

Suppose first that θ is nondecreasing and C is not convex. Then we can find a point $P = \boldsymbol{\alpha}(s_0)$ on the curve and values s_1', s_2' so that $\boldsymbol{\alpha}(s_1')$ and $\boldsymbol{\alpha}(s_2')$ lie on opposite sides of the tangent line to C at P. Then, by the maximum value theorem, there are values s_1 and s_2 so that $\boldsymbol{\alpha}(s_1)$ is the greatest distance "above" the tangent line and $\boldsymbol{\alpha}(s_2)$ is the greatest distance "below." Consider the unit tangent vectors $\mathbf{T}(s_0)$, $\mathbf{T}(s_1)$, and $\mathbf{T}(s_2)$. Since these vectors are either parallel or anti-parallel, some pair must be identical. Letting the respective values of s be s^* and s^{**} with $s^* < s^{**}$, we have $\theta(s^*) = \theta(s^{**})$ (since θ is nondecreasing and $\theta(L) = 2\pi$, the values cannot differ by a multiple of 2π), and therefore $\theta(s) = \theta(s^*)$ for all $s \in [s^*, s^{**}]$. This means that that portion of C between $\boldsymbol{\alpha}(s^*)$ and $\boldsymbol{\alpha}(s^{**})$ is a line segment parallel to the tangent line of C at P; this is a contradiction.

Conversely, suppose C is convex and $\theta(s_1) = \theta(s_2)$ for some $s_1 < s_2$. By Corollary 3.7 there must be s_3 with $\mathbf{T}(s_3) = -\mathbf{T}(s_1) = -\mathbf{T}(s_2)$. Since C is convex, the tangent line at two of $\boldsymbol{\alpha}(s_1)$, $\boldsymbol{\alpha}(s_2)$, and $\boldsymbol{\alpha}(s_3)$ must be the same, say at $\boldsymbol{\alpha}(s^*) = P$ and $\boldsymbol{\alpha}(s^{**}) = Q$. If $\overline{PQ}$ does not lie entirely in C, choose $R \in \overline{PQ}$, $R \notin C$. Since C is convex, the line through R perpendicular to $\overleftrightarrow{PQ}$ must intersect C in at least two points, say M and N, with N farther from $\overleftrightarrow{PQ}$ than M. Since M lies in the interior of $\triangle NPQ$, all three vertices of the triangle can never lie on the same side of any line through M. In particular, N, P, and Q cannot lie on the same side of the tangent line to C at M. Thus, it must be that $\overline{PQ} \subset C$, so $\theta(s) = \theta(s_1) = \theta(s_2)$ for all $s \in [s_1, s_2]$. Therefore, θ is nondecreasing, and we are done. $\square$

Definition. A critical point of κ is called a *vertex* of the curve C.

A closed curve must have at least two vertices: the maximum and minimum points of κ. Every point of a circle is a vertex. We conclude with the following

Proposition 3.9 (Four Vertex Theorem)**.** *A closed convex plane curve has at least four vertices.*

Proof. Suppose that C has fewer than four vertices. As we see from Figure 3.5, either κ must have two critical points (maximum and minimum) or κ must have three critical points (maximum, minimum, and inflection point). More precisely, suppose that κ increases from P to Q and decreases from Q to P. Without loss of generality, we may take P to be at the origin. The equation of $\overleftrightarrow{PQ}$ is $\mathbf{A} \cdot \mathbf{x} = 0$, where we choose $\mathbf{A}$ so that $\kappa'(s) \geq 0$ precisely when $\mathbf{A} \cdot \boldsymbol{\alpha}(s) \geq 0$. Then $\int_C \kappa'(s)(\mathbf{A} \cdot \boldsymbol{\alpha}(s))ds > 0$. Integrating by parts, we have

$$\int_C \kappa'(s)(\mathbf{A} \cdot \boldsymbol{\alpha}(s))ds = -\int_C \kappa(s)(\mathbf{A} \cdot \mathbf{T}(s))ds = \int_C \mathbf{A} \cdot \mathbf{N}'(s)ds = \mathbf{A} \cdot \int_C \mathbf{N}'(s)ds = 0.$$

From this contradiction, we infer that C must have at least four vertices. $\square$

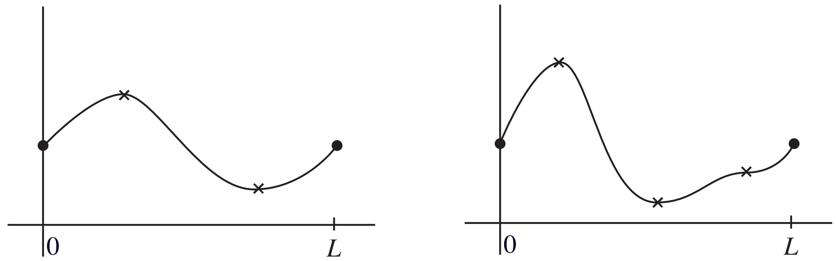

FIGURE 3.5

3.3. The Isoperimetric Inequality. One of the classic questions in mathematics is the following: Given a closed curve of length L, what shape will enclose the most area? A little experimentation will most likely lead the reader to the

Theorem 3.10 (Isoperimetric Inequality)**.** *If a simple closed plane curve C has length L and encloses area A, then*

$$L^2 \geq 4\pi A,$$

and equality holds if and only if C is a circle.

Proof. There are a number of different proofs, but we give one (due to E. Schmidt, 1939) based on Green's Theorem, Theorem 2.6 of the Appendix, and—not surprisingly—relying heavily on the geometric-arithmetic mean inequality and the Cauchy-Schwarz inequality (see Exercise A.1.2). We choose parallel

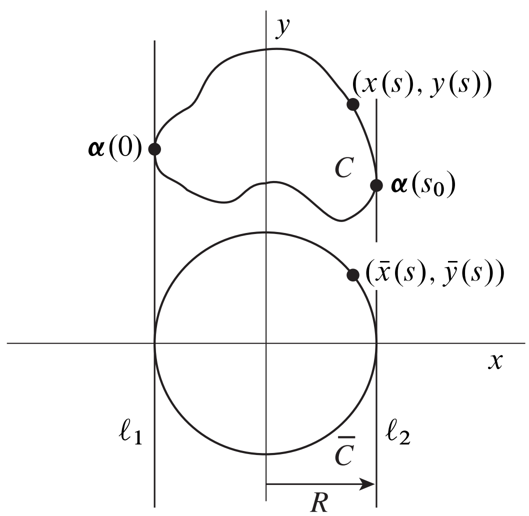

FIGURE 3.6

lines ℓ_1 and ℓ_2 tangent to, and enclosing, C, as pictured in Figure 3.6. We draw a circle $\overline{C}$ of radius R with those same tangent lines and put the origin at its center, with the y-axis parallel to ℓ_i. We now parametrize C by arclength by $\boldsymbol{\alpha}(s) = (x(s), y(s))$, $s \in [0, L]$, taking $\boldsymbol{\alpha}(0) \in \ell_1$ and $\boldsymbol{\alpha}(s_0) \in \ell_2$. We then consider $\overline{\boldsymbol{\alpha}}\colon [0, L] \to \mathbb{R}^2$ given by

$$\overline{\boldsymbol{\alpha}}(s) = \big(\overline{x}(s), \overline{y}(s)\big) = \begin{cases} \big(x(s), -\sqrt{R^2 - x(s)^2}\big), & 0 \leq s \leq s_0 \\ \big(x(s), \sqrt{R^2 - x(s)^2}\big), & s_0 \leq s \leq L \end{cases}.$$

($\overline{\boldsymbol{\alpha}}$ needn't be a parametrization of the circle $\overline{C}$, since it may cover certain portions multiple times, but that's no problem.) Letting A denote the area enclosed by C and $\overline{A} = \pi R^2$ that enclosed by $\overline{C}$, we have (by Exercise A.2.5)

$$A = \int_0^L x(s)y'(s)ds$$

$$\overline{A} = \pi R^2 = -\int_0^L \overline{y}(s)\overline{x}'(s)ds = -\int_0^L \overline{y}(s)x'(s)ds.$$

Adding these equations and applying the Cauchy-Schwarz inequality, we have

$$A + \pi R^2 = \int_0^L \big(x(s)y'(s) - \overline{y}(s)x'(s)\big)ds = \int_0^L \big(x(s), \overline{y}(s)\big) \cdot \big(y'(s), -x'(s)\big)ds$$

$$(*) \qquad \leq \int_0^L \|\big(x(s), \overline{y}(s)\big)\| \|\big(y'(s), -x'(s)\big)\| ds = RL,$$

inasmuch as $\|(y'(s), -x'(s))\| = \|(x'(s), y'(s))\| = 1$ since $\boldsymbol{\alpha}$ is arclength-parametrized. We now recall the arithmetic-geometric mean inequality:

$$\sqrt{ab} \leq \frac{a+b}{2} \qquad \text{for positive numbers } a \text{ and } b,$$

with equality holding if and only if $a = b$. We therefore have

$$\sqrt{A}\sqrt{\pi R^2} \leq \frac{A + \pi R^2}{2} \leq \frac{RL}{2},$$

so $4\pi A \leq L^2$.

Now suppose equality holds here. Then we must have $A = \pi R^2$ and $L = 2\pi R$. It follows that the curve C has the same breadth in all directions (since L now determines R). But equality must also hold in $(*)$, so the vectors $\overline{\boldsymbol{\alpha}}(s) = \big(x(s), \overline{y}(s)\big)$ and $\big(y'(s), -x'(s)\big)$ must be everywhere parallel. Since the first vector has length R and the second has length 1, we infer that

$$\big(x(s), \overline{y}(s)\big) = R\big(y'(s), -x'(s)\big),$$

and so $x(s) = Ry'(s)$. By our remark at the beginning of this paragraph, the same result will hold if we rotate the axes $\pi/2$; let $y = y_0$ be the line halfway between the enclosing *horizontal* lines ℓ_i. Now, substituting $y - y_0$ for x and $-x$ for y, so we have $y(s) - y_0 = -Rx'(s)$, as well. Therefore, $x(s)^2 + \big(y(s) - y_0\big)^2 = R^2(x'(s)^2 + y'(s)^2) = R^2$, and C is indeed a circle of radius R. $\square$

EXERCISES 1.3

1. a. Prove that the shortest path between two points on the unit sphere is an arc of a great circle connecting them. (Hint: Without loss of generality, take one point to be $(0, 0, 1)$ and the other to be $(\sin u_0, 0, \cos u_0)$. Let $\boldsymbol{\alpha}(t) = (\sin u(t) \cos v(t), \sin u(t) \sin v(t), \cos u(t))$, $a \leq t \leq b$, be an arbitrary curve with $u(a) = 0$, $v(a) = 0$, $u(b) = u_0$, $v(b) = 0$, calculate the arclength of $\boldsymbol{\alpha}$, and show that it is smallest when $v(t) = 0$ for all t.)

   b. Prove that if P and Q are points on the unit sphere, then the shortest path between them has length $\arccos(P \cdot Q)$.

2. Give a closed plane curve C with $\kappa > 0$ that is not convex.

3. Draw closed plane curves with rotation indices $0, 2, -2$, and 3, respectively.

*4. Suppose C is a simple closed plane curve with $0 < \kappa \le c$. Prove that length$(C) \ge 2\pi/c$.

5. Give an alternative proof of the latter part of Theorem 3.1 by considering instead the function
$$f(s) = \|\tilde{\mathbf{T}}(s) - \mathbf{T}^*(s)\|^2 + \|\tilde{\mathbf{N}}(s) - \mathbf{N}^*(s)\|^2 + \|\tilde{\mathbf{B}}(s) - \mathbf{B}^*(s)\|^2.$$

6. (See Exercise 1.2.15.) Prove that if C is a simple closed (convex) plane curve of constant breadth μ, then length$(C) = \pi\mu$.

7. A convex plane curve with the origin in its interior can be determined by its tangent lines $(\cos\theta)x + (\sin\theta)y = p(\theta)$, called its *support lines*, as shown in Figure 3.7. The function $p(\theta)$ is called the support function. (Here θ is the polar coordinate, and we assume $p(\theta) > 0$ for all $\theta \in [0, 2\pi]$.)

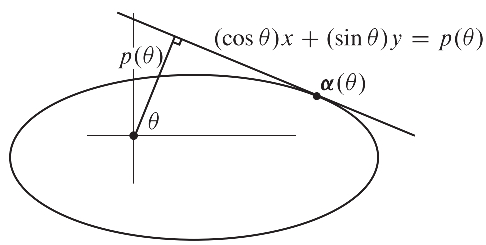

FIGURE 3.7

a. Prove that the line given above is tangent to the curve at the point $\boldsymbol{\alpha}(\theta) = \big(p(\theta)\cos\theta - p'(\theta)\sin\theta,\, p(\theta)\sin\theta + p'(\theta)\cos\theta\big)$.
b. Prove that the curvature of the curve at $\boldsymbol{\alpha}(\theta)$ is $1\big/\big(p(\theta) + p''(\theta)\big)$.
c. Prove that the length of $\boldsymbol{\alpha}$ is given by $L = \displaystyle\int_0^{2\pi} p(\theta)\,d\theta$.
d. Prove that the area enclosed by $\boldsymbol{\alpha}$ is given by $A = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \big(p(\theta)^2 - p'(\theta)^2\big)\,d\theta$.
e. Use the answer to part c to reprove the result of Exercise 6.

8. Let C be a $\mathcal{C}^2$ closed space curve, say parametrized by arclength by $\boldsymbol{\alpha}\colon [0, L] \to \mathbb{R}^3$. A unit *normal field* $\mathbf{X}$ on C is a $\mathcal{C}^1$ vector-valued function with $\mathbf{X}(0) = \mathbf{X}(L)$ and $\mathbf{X}(s)\cdot\mathbf{T}(s) = 0$ and $\|\mathbf{X}(s)\| = 1$ for all s. We define the *twist* of $\mathbf{X}$ to be
$$\mathrm{tw}(C, \mathbf{X}) = \frac{1}{2\pi}\int_0^L \mathbf{X}'(s)\cdot(\mathbf{T}(s)\times\mathbf{X}(s))\,ds.$$

a. Show that if $\mathbf{X}$ and $\mathbf{X}^*$ are two unit normal fields on C, then $\mathrm{tw}(C, \mathbf{X})$ and $\mathrm{tw}(C, \mathbf{X}^*)$ differ by an integer. The fractional part of $\mathrm{tw}(C, \mathbf{X})$ (i.e., the twist mod 1) is called the *total twist* of C. (Hint: Write $\mathbf{X}(s) = \cos\theta(s)\mathbf{N}(s) + \sin\theta(s)\mathbf{B}(s)$.)
b. Prove that the total twist of C equals the fractional part of $\dfrac{1}{2\pi}\displaystyle\int_0^L \tau\,ds$.
c. Prove that if a closed curve lies on a sphere, then its total twist is 0. (Hint: Choose an obvious candidate for $\mathbf{X}$.)

Remark. W. Scherrer proved in 1940 that if the total twist of every closed curve on a surface is 0, then that surface must be a (subset of a) plane or sphere.

9. (See Exercise 1.2.24.) Under what circumstances does a closed space curve have a parallel curve that is also closed? (Hint: Exercise 8 should be relevant.)

10. (*The Bishop Frame*) Suppose $\boldsymbol{\alpha}$ is an arclength-parametrized $\mathcal{C}^2$ curve. Suppose we have $\mathcal{C}^1$ unit vector fields $\mathbf{N}_1$ and $\mathbf{N}_2 = \mathbf{T} \times \mathbf{N}_1$ along $\boldsymbol{\alpha}$ so that
$$\mathbf{T} \cdot \mathbf{N}_1 = \mathbf{T} \cdot \mathbf{N}_2 = \mathbf{N}_1 \cdot \mathbf{N}_2 = 0;$$
i.e., $\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2$ will be a smoothly varying right-handed orthonormal frame as we move along the curve. (To this point, the Frenet frame would work just fine if the curve were $\mathcal{C}^3$ with $\kappa \neq 0$.) But now we want to impose the extra condition that $\mathbf{N}_1' \cdot \mathbf{N}_2 = 0$. We say the unit normal vector field $\mathbf{N}_1$ is *parallel* along $\boldsymbol{\alpha}$; this means that the only change of $\mathbf{N}_1$ is in the direction of $\mathbf{T}$. In this event, $\mathbf{T}, \mathbf{N}_1, \mathbf{N}_2$ is called a *Bishop frame* for $\boldsymbol{\alpha}$. A Bishop frame can be defined even when a Frenet frame cannot (e.g., when there are points with $\kappa = 0$).
   a. Show that there are functions k_1 and k_2 so that
$$\begin{aligned} \mathbf{T}' &= && k_1\mathbf{N}_1 + k_2\mathbf{N}_2 \\ \mathbf{N}_1' &= -k_1\mathbf{T} && \\ \mathbf{N}_2' &= -k_2\mathbf{T} && \end{aligned}$$
   b. Show that $\kappa^2 = k_1^2 + k_2^2$.
   c. Show that if $\boldsymbol{\alpha}$ is $\mathcal{C}^3$ with $\kappa \neq 0$, then we can take $\mathbf{N}_1 = (\cos\theta)\mathbf{N} + (\sin\theta)\mathbf{B}$, where $\theta' = -\tau$. Check that $k_1 = \kappa\cos\theta$ and $k_2 = -\kappa\sin\theta$.
   d. Show that $\boldsymbol{\alpha}$ lies on the surface of a sphere if and only if there are constants λ, μ so that $\lambda k_1 + \mu k_2 + 1 = 0$; moreover, if $\boldsymbol{\alpha}$ lies on a sphere of radius R, then $\lambda^2 + \mu^2 = R^2$. (Cf. Exercise 1.2.19.)
   e. What condition is required to define a Bishop frame globally on a closed curve? (See Exercise 8.) How is this question related to Exercise 1.2.24?

11. Prove Proposition 3.2 as follows. Let $\boldsymbol{\alpha}\colon [0, L] \to \Sigma$ be the arclength parametrization of Γ, and define $\mathbf{F}\colon [0, L] \times [0, 2\pi) \to \Sigma$ by $\mathbf{F}(s, \phi) = \boldsymbol{\xi}$, where $\boldsymbol{\xi}^\perp$ is the great circle making angle ϕ with Γ at $\boldsymbol{\alpha}(s)$. Check that $\mathbf{F}$ takes on the value $\boldsymbol{\xi}$ precisely $\#(\Gamma \cap \boldsymbol{\xi}^\perp)$ times, so that $\mathbf{F}$ is a "multi-parametrization" of Σ that gives us
$$\int_\Sigma \#(\Gamma \cap \boldsymbol{\xi}^\perp)\, d\boldsymbol{\xi} = \int_0^L \int_0^{2\pi} \left\| \frac{\partial \mathbf{F}}{\partial s} \times \frac{\partial \mathbf{F}}{\partial \phi} \right\| d\phi\, ds.$$
Compute that $\left\| \dfrac{\partial \mathbf{F}}{\partial s} \times \dfrac{\partial \mathbf{F}}{\partial \phi} \right\| = |\sin\phi|$ (this is the hard part) and finish the proof. (Hints: As pictured in Figure 3.8, show $\mathbf{v}(s, \phi) = \cos\phi\mathbf{T}(s) + \sin\phi(\boldsymbol{\alpha}(s) \times \mathbf{T}(s))$ is the tangent vector to the great circle $\boldsymbol{\xi}^\perp$ and deduce that $\mathbf{F}(s, \phi) = \boldsymbol{\alpha}(s) \times \mathbf{v}(s, \phi)$. Show that $\dfrac{\partial \mathbf{F}}{\partial \phi}$ and $\boldsymbol{\alpha} \times \dfrac{\partial \mathbf{v}}{\partial s}$ are both multiples of $\mathbf{v}$.)

12. Generalize Theorem 3.5 to prove that if C is a *piecewise-smooth* simple closed plane curve with exterior angles ϵ_j, $j = 1, \dots, \ell$, then $\displaystyle\int_C \kappa\, ds + \sum_{j=1}^{\ell} \epsilon_j = \pm 2\pi$. (As shown in Figure 3.9, the exterior angle ϵ_j

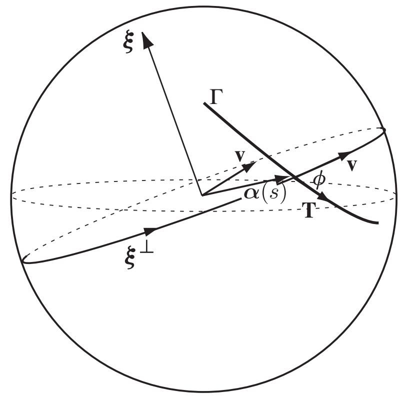

FIGURE 3.8

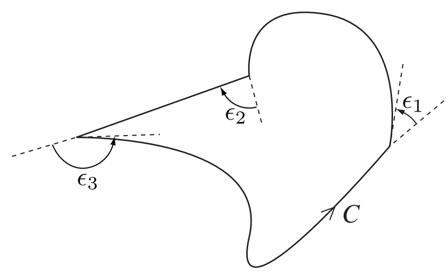

FIGURE 3.9

at $\boldsymbol{\alpha}(s_j)$ is defined to be the angle between $\boldsymbol{\alpha}'_-(s_j) = \lim_{s \to s_j^-} \boldsymbol{\alpha}'(s)$ and $\boldsymbol{\alpha}'_+(s_j) = \lim_{s \to s_j^+} \boldsymbol{\alpha}'(s)$, with the convention that $|\epsilon_j| \leq \pi$.)

13. Complete the details of the proof of the indicated step in the proof of Theorem 3.5, as follows (following H. Hopf's original proof). Pick an interior point $\mathbf{s}_0 \in \Delta$.
    a. Choose $\tilde{\theta}(\mathbf{s}_0)$ so that $\mathbf{h}(\mathbf{s}_0) = \big(\cos\tilde{\theta}(\mathbf{s}_0), \sin\tilde{\theta}(\mathbf{s}_0)\big)$. Use Lemma 3.6, slightly modified, to determine $\tilde{\theta}$ uniquely as a function that is continuous on each ray $\overrightarrow{\mathbf{s}_0\mathbf{s}}$ for every $\mathbf{s} \in \Delta$.
    b. Since a continuous function on a compact (closed and bounded) set $\Delta \subset \mathbb{R}^2$ is uniformly continuous, given any $\varepsilon_0 > 0$, there is a number $\delta_0 > 0$ so that whenever $\mathbf{s}, \mathbf{s}' \in \Delta$ and $\|\mathbf{s} - \mathbf{s}'\| < \delta_0$, we will have $\|\mathbf{h}(\mathbf{s}) - \mathbf{h}(\mathbf{s}')\| < \varepsilon_0$. In particular, show that there is δ_0 so that whenever $\mathbf{s}, \mathbf{s}' \in \Delta$ and $\|\mathbf{s} - \mathbf{s}'\| < \delta_0$, the angle between the vectors $\mathbf{h}(\mathbf{s})$ and $\mathbf{h}(\mathbf{s}')$ is less than π.
    c. Consider the triangle formed by two radii of the unit circle making angle θ. Give an upper bound on θ in terms of the chord length ℓ. Using this, deduce that given $\varepsilon > 0$, there is $0 < \delta < \delta_0$ so that whenever $\|\mathbf{s} - \mathbf{s}'\| < \delta$, we have $|\tilde{\theta}(\mathbf{s}) - \tilde{\theta}(\mathbf{s}') + 2\pi n(\mathbf{s})| < \varepsilon$ for some integer $n(\mathbf{s})$.
    d. Now choose $\mathbf{s}' = \mathbf{s}_1 \in \Delta$ arbitrary. Consider the function $f(u) = \tilde{\theta}(\mathbf{s}_0 + u(\mathbf{s} - \mathbf{s}_0)) - \tilde{\theta}(\mathbf{s}_0 + u(\mathbf{s}_1 - \mathbf{s}_0))$. Show that f is continuous and $f(0) = 0$, and deduce that $|f(1)| < \pi$. Conclude that $n = 0$ in part c and, thus, that $\tilde{\theta}$ is continuous.