$\overrightarrow{QP}$. Express the fact that C moves horizontally to show that $s = -\dfrac{y'(s)}{x'(s)}$; you will need to differentiate unexpectedly. Now use the result of Exercise 4 to find $y = f(x)$. Also see the hint for Exercise 9.)

11. Show that the curve $\boldsymbol{\alpha}(t) = \begin{cases} \big(t, t\sin(\pi/t)\big), & t \neq 0 \\ (0,0), & t = 0 \end{cases}$ has infinite length on $[0,1]$. (Hint: Consider $\ell(\boldsymbol{\alpha}, \mathcal{P}_N)$ with $\mathcal{P}_N = \{0, 1/N, 2/(2N-1), 1/(N-1), \dots, 1/2, 2/3, 1\}$.)

12. Prove that no four distinct points on the twisted cubic (see Example 1(e)) lie on a plane.

13. Consider the "spiral" $\boldsymbol{\alpha}(t) = r(t)(\cos t, \sin t)$, where r is $\mathcal{C}^1$ and $0 \le r(t) \le 1$ for all $t \ge 0$.
    a. Show that if $\boldsymbol{\alpha}$ has finite length on $[0,\infty)$ and r is decreasing, then $r(t) \to 0$ as $t \to \infty$.
    b. Show that if $r(t) = 1/(t+1)$, then $\boldsymbol{\alpha}$ has infinite length on $[0,\infty)$.
    c. If $r(t) = 1/(t+1)^2$, does $\boldsymbol{\alpha}$ have finite length on $[0,\infty)$?
    d. Characterize (in terms of the existence of improper integral(s)) the functions r for which $\boldsymbol{\alpha}$ has finite length on $[0,\infty)$.
    e. Use the result of part d to show that the result of part a holds even without the hypothesis that r be decreasing.

14. (a special case of a recent *American Mathematical Monthly* problem) Suppose $\boldsymbol{\alpha}\colon [a,b] \to \mathbb{R}^2$ is a smooth parametrized plane curve (perhaps not arclength-parametrized). Prove that if the chord length $\|\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(t)\|$ depends only on $|s-t|$, then $\boldsymbol{\alpha}$ must be a (subset of) a line or a circle. (How many derivatives of $\boldsymbol{\alpha}$ do you need to use?)

2. Local Theory: Frenet Frame

What distinguishes a circle or a helix from a line is their *curvature*, i.e., the tendency of the curve to change direction. We shall now see that we can associate to each smooth ($\mathcal{C}^3$) arclength-parametrized curve $\boldsymbol{\alpha}$ a natural "moving frame" (an orthonormal basis for $\mathbb{R}^3$ chosen at each point on the curve, adapted to the geometry of the curve as much as possible).

We begin with a fact from vector calculus that will appear throughout this course.

Lemma 2.1. *Suppose* $\mathbf{f}, \mathbf{g}\colon (a,b) \to \mathbb{R}^3$ *are differentiable and satisfy* $\mathbf{f}(t)\cdot\mathbf{g}(t) = \text{const}$ *for all* t*. Then* $\mathbf{f}'(t)\cdot\mathbf{g}(t) = -\mathbf{f}(t)\cdot\mathbf{g}'(t)$*. In particular,*

$$\boxed{\|\mathbf{f}(t)\| = \text{const} \quad \text{if and only if} \quad \mathbf{f}(t)\cdot\mathbf{f}'(t) = 0 \quad \text{for all } t.}$$

Proof. Since a function is constant on an interval if and only if its derivative is zero everywhere on that interval, we deduce from the product rule,

$$(\mathbf{f}\cdot\mathbf{g})'(t) = \mathbf{f}'(t)\cdot\mathbf{g}(t) + \mathbf{f}(t)\cdot\mathbf{g}'(t),$$

that if $\mathbf{f}\cdot\mathbf{g}$ is constant, then $\mathbf{f}\cdot\mathbf{g}' = -\mathbf{f}'\cdot\mathbf{g}$. In particular, $\|\mathbf{f}\|$ is constant if and only if $\|\mathbf{f}\|^2 = \mathbf{f}\cdot\mathbf{f}$ is constant, and this occurs if and only if $\mathbf{f}\cdot\mathbf{f}' = 0$. $\square$

Remark. This result is intuitively clear. If a particle moves on a sphere centered at the origin, then its velocity vector must be orthogonal to its position vector; any component in the direction of the position vector would move the particle off the sphere. Similarly, suppose $\mathbf{f}$ and $\mathbf{g}$ have constant length and a constant angle between them. Then in order to maintain the constant angle, as $\mathbf{f}$ turns towards $\mathbf{g}$, we see that $\mathbf{g}$ must turn *away* from $\mathbf{f}$ at the same rate.

Using Lemma 2.1 repeatedly, we now construct the *Frenet frame* of suitable regular curves. We assume throughout that the curve $\boldsymbol{\alpha}$ is parametrized by arclength. Then, for starters, $\boldsymbol{\alpha}'(s)$ is the *unit tangent vector* to the curve, which we denote by $\mathbf{T}(s)$. Since $\mathbf{T}$ has constant length, $\mathbf{T}'(s)$ will be orthogonal to $\mathbf{T}(s)$. Assuming $\mathbf{T}'(s) \neq \mathbf{0}$, define the *principal normal vector* $\mathbf{N}(s) = \mathbf{T}'(s)/\|\mathbf{T}'(s)\|$ and the *curvature* $\kappa(s) = \|\mathbf{T}'(s)\|$. So far, we have

$$\mathbf{T}'(s) = \kappa(s)\mathbf{N}(s).$$

If $\kappa(s) = 0$, the principal normal vector is not defined. Assuming $\kappa \neq 0$, we continue. Define the *binormal vector* $\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s)$. Then $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ form a right-handed orthonormal basis for $\mathbb{R}^3$.

Now, $\mathbf{N}'(s)$ must be a linear combination of $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$. But we know from Lemma 2.1 that $\mathbf{N}'(s)\cdot\mathbf{N}(s) = 0$ and $\mathbf{N}'(s)\cdot\mathbf{T}(s) = -\mathbf{T}'(s)\cdot\mathbf{N}(s) = -\kappa(s)$. We define the *torsion* $\tau(s) = \mathbf{N}'(s)\cdot\mathbf{B}(s)$. This gives us

$$\mathbf{N}'(s) = -\kappa(s)\mathbf{T}(s) + \tau(s)\mathbf{B}(s).$$

Finally, $\mathbf{B}'(s)$ must be a linear combination of $\mathbf{T}(s)$, $\mathbf{N}(s)$, and $\mathbf{B}(s)$. Lemma 2.1 tells us that $\mathbf{B}'(s)\cdot\mathbf{B}(s) = 0$, $\mathbf{B}'(s)\cdot\mathbf{T}(s) = -\mathbf{T}'(s)\cdot\mathbf{B}(s) = 0$, and $\mathbf{B}'(s)\cdot\mathbf{N}(s) = -\mathbf{N}'(s)\cdot\mathbf{B}(s) = -\tau(s)$. Thus,

$$\mathbf{B}'(s) = -\tau(s)\mathbf{N}(s).$$

In summary, we have:

Frenet formulas

$$\boxed{\begin{aligned}
\mathbf{T}'(s) &= & \kappa(s)\mathbf{N}(s) & \\
\mathbf{N}'(s) &= -\kappa(s)\mathbf{T}(s) & & + \tau(s)\mathbf{B}(s) \\
\mathbf{B}'(s) &= & -\tau(s)\mathbf{N}(s) &
\end{aligned}}$$

The skew-symmetry of these equations is made clearest when we state the Frenet formulas in matrix form:

$$\begin{bmatrix} | & | & | \\ \mathbf{T}'(s) & \mathbf{N}'(s) & \mathbf{B}'(s) \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{B}(s) \\ | & | & | \end{bmatrix} \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}.$$

Indeed, note that the coefficient matrix appearing on the right is skew-symmetric. This is the case whenever we differentiate an orthogonal matrix depending on a parameter (s in this case). (See Exercise A.1.4.)

Note that, by definition, the curvature, κ, is always nonnegative; the torsion, τ, however, has a sign, as we shall now see.

Example 1. Consider the helix, given by its arclength parametrization (see Exercise 1.1.2) $\boldsymbol{\alpha}(s) = \big(a\cos(s/c), a\sin(s/c), bs/c\big)$, where $c = \sqrt{a^2+b^2}$ and $a > 0$. Then we have

$$\mathbf{T}(s) = \frac{1}{c}\big(-a\sin(s/c), a\cos(s/c), b\big)$$

$$\mathbf{T}'(s) = \frac{1}{c^2}\big(-a\cos(s/c), -a\sin(s/c), 0\big) = \underbrace{\frac{a}{c^2}}_{\kappa(s)} \underbrace{\big(-\cos(s/c), -\sin(s/c), 0\big)}_{\mathbf{N}(s)}.$$

Summarizing,

$$\kappa(s) = \frac{a}{c^2} = \frac{a}{a^2+b^2} \qquad \text{and} \qquad \mathbf{N}(s) = \big(-\cos(s/c), -\sin(s/c), 0\big).$$

Now we deal with $\mathbf{B}$ and the torsion:

$$\mathbf{B}(s) = \mathbf{T}(s) \times \mathbf{N}(s) = \frac{1}{c}\big(b\sin(s/c), -b\cos(s/c), a\big)$$
$$\mathbf{B}'(s) = \frac{1}{c^2}\big(b\cos(s/c), b\sin(s/c), 0\big) = -\frac{b}{c^2}\mathbf{N}(s),$$

so we infer that $\tau(s) = \dfrac{b}{c^2} = \dfrac{b}{a^2+b^2}$.

Note that both the curvature and the torsion are constants. The torsion is positive when the helix is "right-handed" ($b > 0$) and negative when the helix is "left-handed" ($b < 0$). It is interesting to observe that, fixing $a > 0$, as $b \to 0$, the helix becomes very tightly wound and almost planar, and $\tau \to 0$; as $b \to \infty$, the helix twists extremely slowly and looks more and more like a straight line on the cylinder and, once again, $\tau \to 0$. As the reader can check, the helix has the greatest torsion when $b = a$; why does this seem plausible?

In Figure 2.1 we show the Frenet frames of the helix at some sample points. (In the latter two pictures,

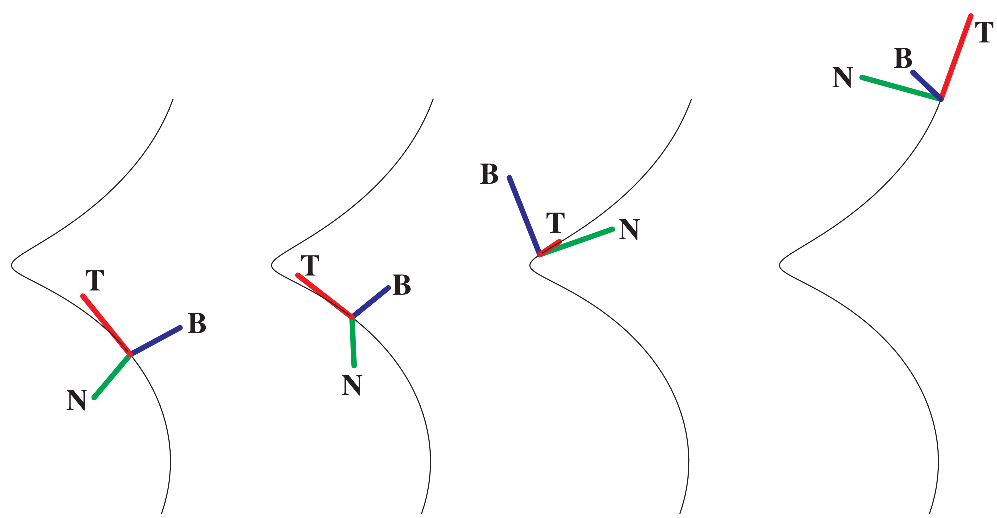

FIGURE 2.1

the perspective is misleading. $\mathbf{T}, \mathbf{N}, \mathbf{B}$ still form a right-handed frame: In the third, $\mathbf{T}$ is in front of $\mathbf{N}$, and in the last, $\mathbf{B}$ is pointing upwards and out of the page.) ∇

We stop for a moment to contemplate what happens with the Frenet formulas when we are dealing with a *non*-arclength-parametrized, regular curve $\boldsymbol{\alpha}$. As we did in Section 1, we can (theoretically) reparametrize by arclength, obtaining $\boldsymbol{\beta}(s)$. Then we have $\boldsymbol{\alpha}(t) = \boldsymbol{\beta}(s(t))$, so, by the chain rule,

$$(*) \qquad \boldsymbol{\alpha}'(t) = \boldsymbol{\beta}'(s(t))s'(t) = \upsilon(t)\mathbf{T}(s(t)),$$

where $\upsilon(t) = s'(t)$ is the speed.[3] Similarly, by the chain rule, once we have the unit tangent vector as a function of t, differentiating with respect to t, we have

$$(\mathbf{T}\circ s)'(t) = \mathbf{T}'(s(t))s'(t) = \upsilon(t)\kappa(s(t))\mathbf{N}(s(t)).$$

Using the more casual—but convenient—Leibniz notation for derivatives,

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds}\frac{ds}{dt} = \upsilon\kappa\mathbf{N} \qquad \text{or} \qquad \kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} = \frac{1}{\upsilon}\frac{d\mathbf{T}}{dt}.$$

Example 2. Let's calculate the curvature of the tractrix (see Example 2 in Section 1). Using the first parametrization, we have $\boldsymbol{\alpha}'(\theta) = (-\sin\theta + \csc\theta, \cos\theta)$, and so

$$\upsilon(\theta) = \|\boldsymbol{\alpha}'(\theta)\| = \sqrt{(-\sin\theta+\csc\theta)^2 + \cos^2\theta} = \sqrt{\csc^2\theta - 1} = -\cot\theta.$$

(Note the negative sign because $\frac{\pi}{2} \le \theta < \pi$.) Therefore,

$$\mathbf{T}(\theta) = -\frac{1}{\cot\theta}(-\sin\theta + \csc\theta, \cos\theta) = -\tan\theta(\cot\theta\cos\theta, \cos\theta) = (-\cos\theta, -\sin\theta).$$

Of course, looking at Figure 1.9, we should expect the formula for $\mathbf{T}$. Then, to find the curvature, we calculate

$$\kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} = \frac{\frac{d\mathbf{T}}{d\theta}}{\frac{ds}{d\theta}} = \frac{(\sin\theta, -\cos\theta)}{-\cot\theta} = (-\tan\theta)(\sin\theta, -\cos\theta).$$

Since $-\tan\theta > 0$ and $(\sin\theta, -\cos\theta)$ is a unit vector we conclude that

$$\kappa(\theta) = -\tan\theta \qquad \text{and} \qquad \mathbf{N}(\theta) = (\sin\theta, -\cos\theta).$$

Later on we will see an interesting geometric consequence of the equality of the curvature and the (absolute value of) the slope. ▽

Example 3. Let's calculate the "Frenet apparatus" for the parametrized curve

$$\boldsymbol{\alpha}(t) = (3t - t^3, 3t^2, 3t + t^3).$$

We begin by calculating $\boldsymbol{\alpha}'$ and determining the unit tangent vector $\mathbf{T}$ and speed υ:

$$\begin{aligned}
\boldsymbol{\alpha}'(t) &= 3(1-t^2, 2t, 1+t^2), \qquad \text{so} \\
\upsilon(t) = \|\boldsymbol{\alpha}'(t)\| &= 3\sqrt{(1-t^2)^2 + (2t)^2 + (1+t^2)^2} = 3\sqrt{2(1+t^2)^2} = 3\sqrt{2}(1+t^2) \qquad \text{and} \\
\mathbf{T}(t) &= \frac{1}{\sqrt{2}}\frac{1}{1+t^2}(1-t^2, 2t, 1+t^2) = \frac{1}{\sqrt{2}}\left(\frac{1-t^2}{1+t^2}, \frac{2t}{1+t^2}, 1\right).
\end{aligned}$$

Now

$$\begin{aligned}
\kappa\mathbf{N} = \frac{d\mathbf{T}}{ds} &= \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} = \frac{1}{\upsilon(t)}\frac{d\mathbf{T}}{dt} \\
&= \frac{1}{3\sqrt{2}(1+t^2)}\frac{1}{\sqrt{2}}\left(\frac{-4t}{(1+t^2)^2}, \frac{2(1-t^2)}{(1+t^2)^2}, 0\right)
\end{aligned}$$

[3] υ is the Greek letter *upsilon*, not to be confused with ν, the Greek letter *nu*.

$$= \underbrace{\frac{1}{3\sqrt{2}(1+t^2)}\frac{1}{\sqrt{2}}\cdot\frac{2}{1+t^2}}_{\kappa}\underbrace{\left(-\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2},0\right)}_{\mathbf{N}}.$$

Here we have factored out the length of the derivative vector and left ourselves with a unit vector in its direction, which must be the principal normal $\mathbf{N}$; the magnitude that is left must be the curvature κ. In summary, so far we have

$$\kappa(t) = \frac{1}{3(1+t^2)^2} \qquad \text{and} \qquad \mathbf{N}(t) = \left(-\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2},0\right).$$

Next we find the binormal $\mathbf{B}$ by calculating the cross product

$$\mathbf{B}(t) = \mathbf{T}(t)\times\mathbf{N}(t) = \frac{1}{\sqrt{2}}\left(-\frac{1-t^2}{1+t^2},-\frac{2t}{1+t^2},1\right).$$

And now, at long last, we calculate the torsion by differentiating $\mathbf{B}$:

$$\begin{aligned}
-\tau\mathbf{N} &= \frac{d\mathbf{B}}{ds} = \frac{\frac{d\mathbf{B}}{dt}}{\frac{ds}{dt}} = \frac{1}{\upsilon(t)}\frac{d\mathbf{B}}{dt} \\
&= \frac{1}{3\sqrt{2}(1+t^2)}\frac{1}{\sqrt{2}}\left(\frac{4t}{(1+t^2)^2},\frac{2(t^2-1)}{(1+t^2)^2},0\right) \\
&= -\underbrace{\frac{1}{3(1+t^2)^2}}_{\tau}\underbrace{\left(-\frac{2t}{1+t^2},\frac{1-t^2}{1+t^2},0\right)}_{\mathbf{N}},
\end{aligned}$$

so $\tau(t) = \kappa(t) = \dfrac{1}{3(1+t^2)^2}$. ∇

Now we see that curvature enters naturally when we compute the acceleration of a moving particle. Differentiating the formula $(*)$ on p. 12, we obtain

$$\begin{aligned}
\boldsymbol{\alpha}''(t) &= \upsilon'(t)\mathbf{T}(s(t)) + \upsilon(t)\mathbf{T}'(s(t))s'(t) \\
&= \upsilon'(t)\mathbf{T}(s(t)) + \upsilon(t)^2\big(\kappa(s(t))\mathbf{N}(s(t))\big).
\end{aligned}$$

Suppressing the variables for a moment, we can rewrite this equation as

$$(**) \qquad \boldsymbol{\alpha}'' = \upsilon'\mathbf{T} + \kappa\upsilon^2\mathbf{N}.$$

The tangential component of acceleration is the derivative of speed; the normal component (the "centripetal acceleration" in the case of circular motion) is the product of the curvature of the path *and* the square of the speed. Thus, from the physics of the motion we can recover the curvature of the path:

Proposition 2.2. *For any regular parametrized curve* $\boldsymbol{\alpha}$, *we have* $\kappa = \dfrac{\|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}$.

Proof. Since $\boldsymbol{\alpha}'\times\boldsymbol{\alpha}'' = (\upsilon\mathbf{T})\times(\upsilon'\mathbf{T}+\kappa\upsilon^2\mathbf{N}) = \kappa\upsilon^3\mathbf{T}\times\mathbf{N}$ and $\kappa\upsilon^3 > 0$, we obtain $\kappa\upsilon^3 = \|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''\|$, and so $\kappa = \|\boldsymbol{\alpha}'\times\boldsymbol{\alpha}''\|/\upsilon^3$, as desired. $\square$

We next proceed to study various theoretical consequences of the Frenet formulas.

Proposition 2.3. *A space curve is a line if and only if its curvature is everywhere* 0.

Proof. The general line is given by $\boldsymbol{\alpha}(s) = s\mathbf{v} + \mathbf{c}$ for some unit vector $\mathbf{v}$ and constant vector $\mathbf{c}$. Then $\boldsymbol{\alpha}'(s) = \mathbf{T}(s) = \mathbf{v}$ is constant, so $\kappa = 0$. Conversely, if $\kappa = 0$, then $\mathbf{T}(s) = \mathbf{T_0}$ is a constant vector, and, integrating, we obtain $\boldsymbol{\alpha}(s) = \int_0^s \mathbf{T}(u)du + \boldsymbol{\alpha}(0) = s\mathbf{T_0} + \boldsymbol{\alpha}(0)$. This is, once again, the parametric equation of a line. $\square$

Example 4. Suppose all the tangent lines of a space curve pass through a fixed point. What can we say about the curve? Without loss of generality, we take the fixed point to be the origin and the curve to be arclength-parametrized by $\boldsymbol{\alpha}$. Then there is a scalar function λ so that for every s we have $\boldsymbol{\alpha}(s) = \lambda(s)\mathbf{T}(s)$. Differentiating, we have

$$\mathbf{T}(s) = \boldsymbol{\alpha}'(s) = \lambda'(s)\mathbf{T}(s) + \lambda(s)\mathbf{T}'(s) = \lambda'(s)\mathbf{T}(s) + \lambda(s)\kappa(s)\mathbf{N}(s).$$

Then $(\lambda'(s) - 1)\mathbf{T}(s) + \lambda(s)\kappa(s)\mathbf{N}(s) = \mathbf{0}$, so, since $\mathbf{T}(s)$ and $\mathbf{N}(s)$ are linearly independent, we infer that $\lambda(s) = s + c$ for some constant c and $\kappa(s) = 0$. Therefore, the curve must be a line through the fixed point. ∇

Somewhat more challenging is the following

Proposition 2.4. *A space curve is planar if and only if its torsion is everywhere* 0. *The only planar curves with nonzero constant curvature are (portions of) circles.*

Proof. If a curve lies in a plane $\mathcal{P}$, then $\mathbf{T}(s)$ and $\mathbf{N}(s)$ span the plane $\mathcal{P}_0$ parallel to $\mathcal{P}$ and passing through the origin. Therefore, $\mathbf{B} = \mathbf{T} \times \mathbf{N}$ is a constant vector (the normal to $\mathcal{P}_0$), and so $\mathbf{B}' = -\tau\mathbf{N} = \mathbf{0}$, from which we conclude that $\tau = 0$. Conversely, if $\tau = 0$, the binormal vector $\mathbf{B}$ is a constant vector $\mathbf{B_0}$. Now, consider the function $f(s) = \boldsymbol{\alpha}(s) \cdot \mathbf{B_0}$; we have $f'(s) = \boldsymbol{\alpha}'(s) \cdot \mathbf{B_0} = \mathbf{T}(s) \cdot \mathbf{B}(s) = 0$, and so $f(s) = c$ for some constant c. This means that $\boldsymbol{\alpha}$ lies in the plane $\mathbf{x} \cdot \mathbf{B_0} = c$.

We leave it to the reader to check in Exercise 2a. that a circle of radius a has constant curvature $1/a$. (This can also be deduced as a special case of the calculation in Example 1.) Now suppose a planar curve $\boldsymbol{\alpha}$ has constant curvature κ_0. Consider the auxiliary function $\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(s) + \dfrac{1}{\kappa_0}\mathbf{N}(s)$. Then we have $\boldsymbol{\beta}'(s) = \boldsymbol{\alpha}'(s) + \dfrac{1}{\kappa_0}(-\kappa_0(s)\mathbf{T}(s)) = \mathbf{T}(s) - \mathbf{T}(s) = \mathbf{0}$. Therefore $\boldsymbol{\beta}$ is a constant function, say $\boldsymbol{\beta}(s) = P$ for all s. Now we claim that $\boldsymbol{\alpha}$ is a (subset of a) circle centered at P, for $\|\boldsymbol{\alpha}(s) - P\| = \|\boldsymbol{\alpha}(s) - \boldsymbol{\beta}(s)\| = 1/\kappa_0$. $\square$

We have already seen that a circular helix has constant curvature and torsion. We leave it to the reader to check in Exercise 10 that these are the only curves with constant curvature and torsion. Somewhat more interesting are the curves for which τ/κ is a constant.

A *generalized helix* is a space curve with $\kappa \neq 0$ all of whose tangent vectors make a constant angle with a fixed direction. As shown in Figure 2.2, this curve lies on a generalized cylinder, formed by taking the union of the lines (rulings) in that fixed direction through each point of the curve. We can now characterize generalized helices by the following

Proposition 2.5. *A curve is a generalized helix if and only if* τ/κ *is constant.*

Proof. Suppose $\boldsymbol{\alpha}$ is an arclength-parametrized generalized helix. Then there is a (constant) unit vector $\mathbf{A}$ with the property that $\mathbf{T} \cdot \mathbf{A} = \cos\theta$ for some constant θ. Differentiating, we obtain $\kappa\mathbf{N} \cdot \mathbf{A} = 0$, whence $\mathbf{N} \cdot \mathbf{A} = 0$. Differentiating yet again, we have

$$(-\kappa\mathbf{T} + \tau\mathbf{B}) \cdot \mathbf{A} = 0. \tag{†}$$

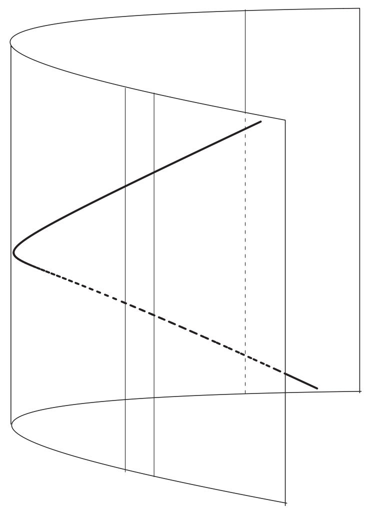

FIGURE 2.2

Now, note that $\mathbf{A}$ lies in the plane spanned by $\mathbf{T}$ and $\mathbf{B}$, and thus $\mathbf{B} \cdot \mathbf{A} = \pm\sin\theta$. Thus, we infer from equation (†) that $\tau/\kappa = \pm\cot\theta$, which is indeed constant.

Conversely, if τ/κ is constant, set $\tau/\kappa = \cot\theta$ for some angle $\theta \in (0, \pi)$. Set $\mathbf{A}(s) = \cos\theta\mathbf{T}(s) + \sin\theta\mathbf{B}(s)$. Then $\mathbf{A}'(s) = (\kappa\cos\theta - \tau\sin\theta)\mathbf{N}(s) = \mathbf{0}$, so $\mathbf{A}(s)$ is a constant unit vector $\mathbf{A}$, and $\mathbf{T}(s) \cdot \mathbf{A} = \cos\theta$ is constant, as desired. $\square$

Example 5. In Example 3 we saw a curve $\boldsymbol{\alpha}$ with $\kappa = \tau$, so from the proof of Proposition 2.5 we see that the curve should make a constant angle $\theta = \pi/4$ with the vector $\mathbf{A} = \dfrac{1}{\sqrt{2}}(\mathbf{T} + \mathbf{B}) = (0, 0, 1)$ (as should have been obvious from the formula for $\mathbf{T}$ alone). We verify this in Figure 2.3 by drawing $\boldsymbol{\alpha}$ along with the vertical cylinder built on the projection of $\boldsymbol{\alpha}$ onto the xy-plane. ∇

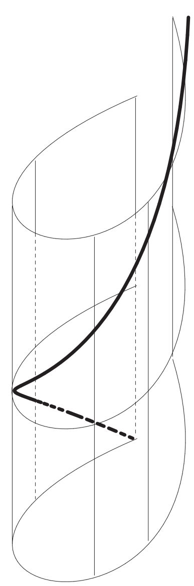

FIGURE 2.3

The Frenet formulas actually characterize the local picture of a space curve.

Proposition 2.6 (Local canonical form)**.** *Let* $\boldsymbol{\alpha}$ *be a smooth* ($\mathcal{C}^3$ *or better) arclength-parametrized curve. If* $\boldsymbol{\alpha}(0) = \mathbf{0}$, *then for* s *near* 0, *we have*

$$\boldsymbol{\alpha}(s) = \left(s - \frac{\kappa_0^2}{6}s^3 + \dots\right)\mathbf{T}(0) + \left(\frac{\kappa_0}{2}s^2 + \frac{\kappa_0'}{6}s^3 + \dots\right)\mathbf{N}(0) + \left(\frac{\kappa_0\tau_0}{6}s^3 + \dots\right)\mathbf{B}(0).$$

(Here κ_0, τ_0, *and* κ_0' *denote, respectively, the values of* κ, τ, *and* κ' *at* 0, *and* $\lim\limits_{s\to 0} \dots/s^3 = 0$.*)*

Proof. Using Taylor's Theorem, we write

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}(0) + s\boldsymbol{\alpha}'(0) + \frac{1}{2}s^2\boldsymbol{\alpha}''(0) + \frac{1}{6}s^3\boldsymbol{\alpha}'''(0) + \dots,$$

where $\lim\limits_{s\to 0} \dots/s^3 = 0$. Now, $\boldsymbol{\alpha}(0) = \mathbf{0}$, $\boldsymbol{\alpha}'(0) = \mathbf{T}(0)$, and $\boldsymbol{\alpha}''(0) = \mathbf{T}'(0) = \kappa_0\mathbf{N}(0)$. Differentiating again, we have $\boldsymbol{\alpha}'''(0) = (\kappa\mathbf{N})'(0) = \kappa_0'\mathbf{N}(0) + \kappa_0(-\kappa_0\mathbf{T}(0) + \tau_0\mathbf{B}(0))$. Substituting, we obtain

$$\begin{aligned}\boldsymbol{\alpha}(s) &= s\mathbf{T}(0) + \frac{1}{2}s^2\kappa_0\mathbf{N}(0) + \frac{1}{6}s^3\left(-\kappa_0^2\mathbf{T}(0) + \kappa_0'\mathbf{N}(0) + \kappa_0\tau_0\mathbf{B}(0)\right) + \dots \\ &= \left(s - \frac{\kappa_0^2}{6}s^3 + \dots\right)\mathbf{T}(0) + \left(\frac{\kappa_0}{2}s^2 + \frac{\kappa_0'}{6}s^3 + \dots\right)\mathbf{N}(0) + \left(\frac{\kappa_0\tau_0}{6}s^3 + \dots\right)\mathbf{B}(0),\end{aligned}$$

as required. □

We now introduce three fundamental planes at $P = \boldsymbol{\alpha}(0)$:

(i) the *osculating plane*, spanned by $\mathbf{T}(0)$ and $\mathbf{N}(0)$,
(ii) the *rectifying plane*, spanned by $\mathbf{T}(0)$ and $\mathbf{B}(0)$, and
(iii) the *normal plane*, spanned by $\mathbf{N}(0)$ and $\mathbf{B}(0)$.

We see that, locally, the projections of $\boldsymbol{\alpha}$ into these respective planes look like

(i) $\left(u - (\kappa_0^2/6)u^3 + \dots, (\kappa_0/2)u^2 + (\kappa_0'/6)u^3 + \dots\right)$
(ii) $\left(u - (\kappa_0^2/6)u^3 + \dots, (\kappa_0\tau_0/6)u^3 + \dots\right)$, and
(iii) $\left((\kappa_0/2)u^2 + (\kappa_0'/6)u^3 + \dots, (\kappa_0\tau_0/6)u^3 + \dots\right)$,

where $\lim\limits_{u\to 0} \dots/u^3 = 0$. Thus, the projections of $\boldsymbol{\alpha}$ into these planes look locally as shown in Figure 2.4. The osculating ("kissing") plane is the plane that comes closest to containing $\boldsymbol{\alpha}$ near P (see also Exercise

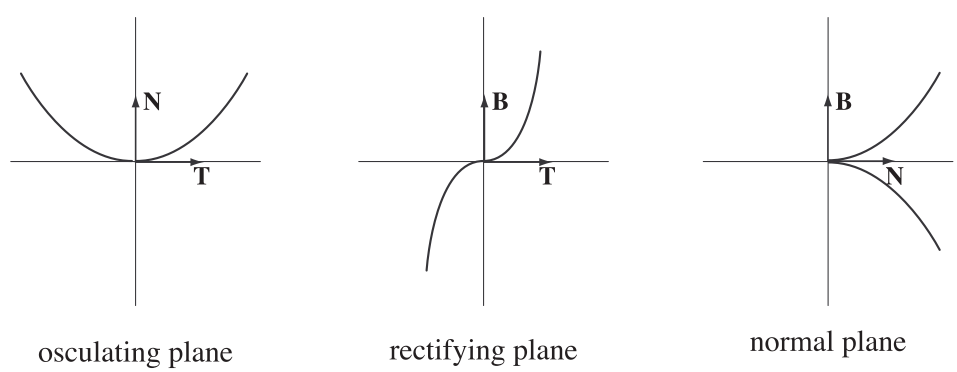

FIGURE 2.4

25); the rectifying ("straightening") plane is the one that comes closest to flattening the curve near P; the normal plane is normal (perpendicular) to the curve at P. (Cf. Figure 1.3.)

EXERCISES 1.2

1. Compute the curvature of the following arclength-parametrized curves:
   a. $\boldsymbol{\alpha}(s) = \left(\frac{1}{\sqrt{2}}\cos s, \frac{1}{\sqrt{2}}\cos s, \sin s\right)$
   b. $\boldsymbol{\alpha}(s) = \left(\sqrt{1+s^2}, \ln(s+\sqrt{1+s^2})\right)$
   *c. $\boldsymbol{\alpha}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right), s \in (-1,1)$
2. Calculate the unit tangent vector, principal normal, and curvature of the following curves:
   a. a circle of radius a: $\boldsymbol{\alpha}(t) = (a\cos t, a\sin t)$
   b. $\boldsymbol{\alpha}(t) = (t, \cosh t)$
   c. $\boldsymbol{\alpha}(t) = (\cos^3 t, \sin^3 t), t \in (0, \pi/2)$
3. Calculate the Frenet apparatus ($\mathbf{T}, \kappa, \mathbf{N}, \mathbf{B}$, and τ) of the following curves:
   *a. $\boldsymbol{\alpha}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right), s \in (-1,1)$
   b. $\boldsymbol{\alpha}(t) = \left(\frac{1}{2}e^t(\sin t + \cos t), \frac{1}{2}e^t(\sin t - \cos t), e^t\right)$
   *c. $\boldsymbol{\alpha}(t) = \left(\sqrt{1+t^2}, t, \ln(t+\sqrt{1+t^2})\right)$
   d. $\boldsymbol{\alpha}(t) = (e^t\cos t, e^t\sin t, e^t)$
   e. $\boldsymbol{\alpha}(t) = (\cosh t, \sinh t, t)$
   f. $\boldsymbol{\alpha}(t) = \left(t, t^2/2, t\sqrt{1+t^2} + \ln(t+\sqrt{1+t^2})\right)$
   g. $\boldsymbol{\alpha}(t) = (t - \sin t\cos t, \sin^2 t, \cos t), t \in (0, \pi)$

♯4. Prove that the curvature of the plane curve $y = f(x)$ is given by $\kappa = \dfrac{|f''|}{(1+f'^2)^{3/2}}$.

♯*5. Use Proposition 2.2 and the second parametrization of the tractrix given in Example 2 of Section 1 to recompute the curvature.

*6. By differentiating the equation $\mathbf{B} = \mathbf{T} \times \mathbf{N}$, derive the equation $\mathbf{B}' = -\tau\mathbf{N}$.

♯7. Suppose $\boldsymbol{\alpha}$ is an arclength-parametrized space curve with the property that $\|\boldsymbol{\alpha}(s)\| \le \|\boldsymbol{\alpha}(s_0)\| = R$ for all s sufficiently close to s_0. Prove that $\kappa(s_0) \ge 1/R$. (Hint: Consider the function $f(s) = \|\boldsymbol{\alpha}(s)\|^2$. What do you know about $f''(s_0)$?)

8. Let $\boldsymbol{\alpha}$ be a regular (arclength-parametrized) curve with nonzero curvature. The normal line to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(s)$ is the line through $\boldsymbol{\alpha}(s)$ with direction vector $\mathbf{N}(s)$. Suppose all the normal lines to $\boldsymbol{\alpha}$ pass through a fixed point. What can you say about the curve?

9. a. Prove that if all the normal planes of a curve pass through a particular point, then the curve lies on a sphere. (Hint: Apply Lemma 2.1.)
   *b. Prove that if all the osculating planes of a curve pass through a particular point, then the curve is planar.

10. Prove that if $\kappa = \kappa_0$ and $\tau = \tau_0$ are nonzero constants, then the curve is a (right) circular helix.