Chapter 5

Comparing Graphs

5.1 Overview

It is rare than one can analytically determine the eigenvalues of an abstractly defined graph. Usually one is only able to prove loose bounds on some eigenvalues.

In this lecture we will see a powerful technique that allows one to compare one graph with another, and prove things like lower bounds on the smallest eigenvalue of a Laplacians. It often goes by the name "Poincaré Inequalities" (see [DS91, SJ89, GLM99]), or "Graphic inequalities".

5.2 The Loewner order

I begin by recalling an extremely useful piece of notation that is used in the Optimization community. For a symmetric matrix $\boldsymbol{A}$, we write

$$\boldsymbol{A} \succcurlyeq 0$$

if $\boldsymbol{A}$ is *positive semidefinite*. That is, if all of the eigenvalues of $\boldsymbol{A}$ are nonnegative, which is equivalent to

$$\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} \geq \boldsymbol{0},$$

for all $\boldsymbol{v}$. We similarly write

$$\boldsymbol{A} \succcurlyeq \boldsymbol{B}$$

if

$$\boldsymbol{A} - \boldsymbol{B} \succcurlyeq \boldsymbol{0}$$

which is equivalent to

$$\boldsymbol{v}^T \boldsymbol{A} \boldsymbol{v} \geq \boldsymbol{v}^T \boldsymbol{B} \boldsymbol{v}$$

for all $\boldsymbol{v}$.

The relation $\succcurlyeq$ is called the Loewner *partial order*. It applies to some pairs of symmetric matrices, while others are incomparable. But, for all pairs to which it does apply, it acts like an order. For example, we have

$$\boldsymbol{A} \succcurlyeq \boldsymbol{B} \text{ and } \boldsymbol{B} \succcurlyeq \boldsymbol{C} \text{ implies } \boldsymbol{A} \succcurlyeq \boldsymbol{C},$$

and

$$\boldsymbol{A} \succcurlyeq \boldsymbol{B} \text{ implies } \boldsymbol{A} + \boldsymbol{C} \succcurlyeq \boldsymbol{B} + \boldsymbol{C},$$

for symmetric matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$.

We will overload this notation by defining it for graphs as well. Thus, we write

$$G \succcurlyeq H$$

if $\boldsymbol{L}_G \succcurlyeq \boldsymbol{L}_H$. When we write this, we are always describing an inequality on Laplacian matrices.

For example, if $G = (V, E)$ is a graph and $H = (V, F)$ is a subgraph of G, then

$$\boldsymbol{L}_G \succcurlyeq \boldsymbol{L}_H.$$

To see this, recall the Laplacian quadratic form:

$$\boldsymbol{x}^T \boldsymbol{L}_G \boldsymbol{x} = \sum_{(u,v) \in E} w_{u,v} (\boldsymbol{x}(u) - \boldsymbol{x}(v))^2.$$

It is clear that dropping edges can only decrease the value of the quadratic form. The same holds for decreasing the weights of edges.

This notation is particularly useful when we consider some multiple of a graph, such as when we write

$$G \succcurlyeq c \cdot H,$$

for some $c > 0$. What is $c \cdot H$? It is the same graph as H, but the weight of every edge is multiplied by c.

We usually use this notation for the inequalities it implies on the eigenvalues of $\boldsymbol{L}_G$ and $\boldsymbol{L}_H$.

Lemma 5.2.1. *If G and H are graphs such that*

$$G \succcurlyeq c \cdot H,$$

then

$$\lambda_k(G) \geq c\lambda_k(H),$$

for all k.

Proof. The Courant-Fischer Theorem tells us that

$$\lambda_k(G) = \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = k}} \max_{\boldsymbol{x} \in S} \frac{\boldsymbol{x}^T L_G \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} \geq c \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = k}} \max_{\boldsymbol{x} \in S} \frac{\boldsymbol{x}^T L_H \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = c\lambda_k(H).$$

□

Corollary 5.2.2. *Let G be a graph and let H be obtained by either adding an edge to G or increasing the weight of an edge in G. Then, for all a*

$$\lambda_i(G) \leq \lambda_i(H).$$

5.3 Approximations of Graphs

We consider one graph to be a good approximation of another if their Laplacian quadratic forms are similar. For example, we will say that H is a c-approximation of G if

$$cH \succcurlyeq G \succcurlyeq H/c.$$

Surprising approximations exist. For example, random regular and random Erdös-Rényi graphs are good approximations of complete graphs. We will encounter infinite families of expander graphs that for every $\epsilon > 0$ provide a $d > 0$ such that for all $n > 0$ there is a d-regular graph G_n that is a $(1+\epsilon)$-approximation of K_n. As d is fixed, such a graph has many fewer edges than a complete graph!

In Chapters **??** and **??** we will also prove that every graph can be well-approximated by a sparse graph.

5.4 The Path Inequality

By now you should be wondering, "how do we prove that $G \succcurlyeq c \cdot H$ for some graph G and H?" Not too many ways are known. We'll do it by proving some inequalities of this form for some of the simplest graphs, and then extending them to more general graphs. For example, we will prove

$$(n-1) \cdot P_n \succcurlyeq G_{1,n}, \tag{5.1}$$

where P_n is the path from vertex 1 to vertex n, and $G_{1,n}$ is the graph with just the edge $(1,n)$. All of these edges are unweighted.

The following very simple proof of this inequality was discovered by Sam Daitch.

Lemma 5.4.1.

$$(n-1) \cdot P_n \succcurlyeq G_{1,n}.$$

Proof. We need to show that for every $\boldsymbol{x} \in \mathbb{R}^n$,

$$(n-1)\sum_{a=1}^{n-1}(\boldsymbol{x}(a+1)-\boldsymbol{x}(a))^2 \geq (\boldsymbol{x}(n)-\boldsymbol{x}(1))^2.$$

For $1 \leq a \leq n-1$, set

$$\boldsymbol{\Delta}(a) = \boldsymbol{x}(a+1) - \boldsymbol{x}(a).$$

The inequality we need to prove then becomes

$$(n-1)\sum_{a=1}^{n-1}\boldsymbol{\Delta}(a)^2 \geq \left(\sum_{a=1}^{n-1}\boldsymbol{\Delta}(a)\right)^2.$$

But, this is just the Cauchy-Schwartz inequality. I'll remind you that Cauchy-Schwartz follows from the fact that the inner product of two vectors is at most the product of their norms. In this case, those vectors are $\boldsymbol{\Delta}$ and the all-ones vector of length $n-1$:

$$\left(\sum_{a=1}^{n-1} \boldsymbol{\Delta}(a)\right)^2 = \left(\mathbf{1}_{n-1}^T \boldsymbol{\Delta}\right)^2 \leq \left(\|\mathbf{1}_{n-1}\| \, \|\boldsymbol{\Delta}\|\right)^2 = \|\mathbf{1}_{n-1}\|^2 \, \|\boldsymbol{\Delta}\|^2 = (n-1) \sum_{i=1}^{n-1} \boldsymbol{\Delta}(i)^2.$$

□

5.4.1 Bounding λ_2 of a Path Graph

In Lemma 6.6.1 we will prove that $\lambda_2(P_n) \approx \pi^2/n^2$. For now, we demonstrate the power of Lemma 5.4.1 by using it to prove a lower bound on $\lambda_2(P_n)$ that is very close to this.

To prove a lower bound on $\lambda_2(P_n)$, we will prove that some multiple of the path is at least the complete graph. To this end, write

$$L_{K_n} = \sum_{a<b} L_{G_{a,b}},$$

and recall that

$$\lambda_2(K_n) = n.$$

For every $a < b$, let $P_{a,b}$ be the subgraph of the path graph induced on vertices with indices between a and b. Note that this is itself a path graph of length $b - a$.

For every edge (a, b) in the complete graph, we apply the only inequality available in the path:

$$G_{a,b} \preccurlyeq (b-a) P_{a,b} \preccurlyeq (b-a) P_n. \tag{5.2}$$

This inequality says that $G_{a,b}$ is at most $(b-a)$ times the part of the path connecting a to b, and that this part of the path is less than the whole.

Summing inequality (5.2) over all edges $(a, b) \in K_n$ gives

$$K_n = \sum_{a,b} G_{a,b} \preccurlyeq \sum_{a,b} (b-a) P_n.$$

To finish the proof, we compute

$$\sum_{1 \leq a < b \leq n} (b-a) = \sum_{c=1}^{n-1} c(n-c) = n(n+1)(n-1)/6.$$

So,

$$L_{K_n} \preccurlyeq \frac{n(n+1)(n-1)}{6} L_{P_n}.$$

Applying Lemma 5.2.1, we obtain

$$\frac{6}{(n+1)(n-1)} \leq \lambda_2(P_n).$$

5.5 The Complete Binary Tree

Let's do the same analysis with the complete binary tree.

One way of understanding the complete binary tree of depth $d+1$ is to identify the vertices of the tree with strings over $\{0,1\}$ of length at most d. The root of the tree is the empty string. Every other node has one ancestor, which is obtained by removing the last character of its string, and two children, which are obtained by appending one character to its label.

Alternatively, you can describe it as the graph on $n = 2^{d+1} - 1$ nodes with edges of the form $(i, 2i)$ and $(i, 2i+1)$ for $i < n$. We will name this graph T_d. See figure 5.1 for pictures of these.

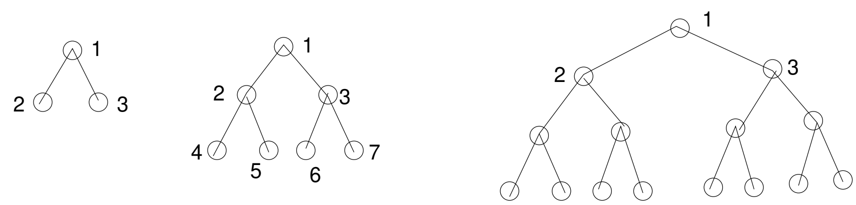

Figure 5.1: T_1, T_2 and T_3. Node 1 is at the top, 2 and 3 are its children. Some other nodes have been labeled as well.

Let's first upper bound $\lambda_2(T_d)$ by constructing a test vector $\boldsymbol{x}$. Set $\boldsymbol{x}(1) = 0$, $\boldsymbol{x}(2) = 1$, and $\boldsymbol{x}(3) = -1$. Then, for every vertex u that we can reach from node 2 without going through node 1, we set $\boldsymbol{x}(a) = 1$. For all the other nodes, we set $\boldsymbol{x}(a) = -1$.

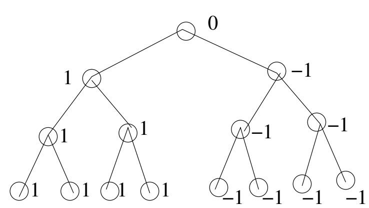

Figure 5.2: The test vector we use to upper bound $\lambda_2(T_3)$.

We have constructed $\boldsymbol{x}$ symmetrically, so that $\mathbf{1}^T \boldsymbol{x} = 0$. Thus,

$$\lambda_2 \leq \frac{\boldsymbol{x}^T \boldsymbol{L} \boldsymbol{x}}{\boldsymbol{x}^T \boldsymbol{x}} = \frac{\sum_{a \sim b} (\boldsymbol{x}(a) - \boldsymbol{x}(b))^2}{\sum_i \boldsymbol{x}(a)^2} = \frac{(\boldsymbol{x}(1) - \boldsymbol{x}(2))^2 + (\boldsymbol{x}(1) - \boldsymbol{x}(3))^2}{n-1} = 2/(n-1).$$

We will again prove a lower bound by comparing T_d to the complete graph. For each edge $a < b$, let $T_d^{a,b}$ denote the unique path in T from a to b. This path will have length at most $2d \leq 2 \log_2 n$. So, we have

$$K_n = \sum_{a<b} G_{a,b} \preccurlyeq \sum_{a<b} (2d) T_d^{a,b} \preccurlyeq \sum_{a<b} (2 \log_2 n) T_d = \binom{n}{2} (2 \log_2 n) T_d.$$

So, we obtain the bound

$$\binom{n}{2} (2 \log_2 n) \lambda_2(T_d) \geq n,$$

which implies

$$\lambda_2(T_d) \geq \frac{1}{(n-1)\log_2 n}.$$

Using the generalization of Lemma 5.4.1 presented in the next section, one can improve this lower bound to $1/cn$ for some constant c.

5.6 The weighted path

We now generalize the the inequality in Lemma 5.4.1 to weighted path graphs. Allowing for weights on the edges of the path greatly extends it applicability.

Lemma 5.6.1. *Let $w_1, \ldots, w_{n-1}$ be positive. Then*

$$G_{1,n} \preccurlyeq \left(\sum_{i=1}^{n-1} \frac{1}{w_a}\right) \sum_{a=1}^{n-1} w_a G_{a,a+1}.$$

Proof. Let $\boldsymbol{x} \in \mathbb{R}^n$ and set $\boldsymbol{\Delta}(a)$ as in the proof of Lemma 5.4.1 Now, set

$$\boldsymbol{\gamma}(a) = \boldsymbol{\Delta}(a)\sqrt{w_a}.$$

Let $\boldsymbol{w}^{-1/2}$ denote the vector for which

$$\boldsymbol{w}^{-1/2}(a) = \frac{1}{\sqrt{w_a}}.$$

Then,

$$\sum_a \boldsymbol{\Delta}(a) = \boldsymbol{\gamma}^T \boldsymbol{w}^{-1/2},$$

$$\left\| \boldsymbol{w}^{-1/2} \right\|^2 = \sum_a \frac{1}{w_a},$$

and

$$\|\boldsymbol{\gamma}\|^2 = \sum_a \boldsymbol{\Delta}(a)^2 w_a.$$

So,

$$\begin{aligned}
\boldsymbol{x}^T L_{G_{1,n}} \boldsymbol{x} &= \left(\sum_a \boldsymbol{\Delta}(a)\right)^2 = \left(\boldsymbol{\gamma}^T \boldsymbol{w}^{-1/2}\right)^2 \\
&\leq \left(\|\boldsymbol{\gamma}\| \left\|\boldsymbol{w}^{-1/2}\right\|\right)^2 = \left(\sum_a \frac{1}{w_a}\right) \sum_a \boldsymbol{\Delta}(a)^2 w_a = \left(\sum_a \frac{1}{w_a}\right) \boldsymbol{x}^T \left(\sum_{a=1}^{n-1} w_a L_{G_{a,a+1}}\right) \boldsymbol{x}.
\end{aligned}$$

□

5.7 Exercises

1. Let $\boldsymbol{v}$ be a vector so that $\boldsymbol{v}^T\mathbf{1} = 0$. Prove that

$$\|\boldsymbol{v}\|^2 \leq \|\boldsymbol{v} + t\mathbf{1}\|^2 ,$$

for every real number t.