Quadratic Forms and Graph Laplacian

In the minihomework, you proved that

$$\langle \vec{x}, A\vec{y} \rangle = \langle A^T\vec{x}, \vec{y} \rangle$$

for any vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ and $n \times n$ matrix A.

Notice that if A is symmetric, then

$$\langle \vec{x}, A\vec{y} \rangle = \langle A\vec{x}, \vec{y} \rangle$$

Definition. If A is an $n \times n$ symmetric matrix, it has an associated <u>quadratic form</u> $Q_A : \mathbb{R}^n \to \mathbb{R}$ defined by

$$Q_A(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle$$

Proposition. A quadratic form Q_A has

$$Q_A(\lambda \vec{x}) = \lambda^2 Q_A(x)$$

and

$$\langle \vec{x}, \vec{y} \rangle_A := \frac{1}{2}\left(Q_A(\vec{x}+\vec{y}) - Q_A(\vec{x}) - Q_A(\vec{y})\right)$$

bilinear in $\vec{x}$ and $\vec{y}$.

Proof. We know $\langle \lambda\vec{x}, A\lambda\vec{x} \rangle = \lambda^2 \langle \vec{x}, A\vec{x} \rangle$ because matrix multiplication is linear (so $A\lambda\vec{x} = \lambda A\vec{x}$) and the dot product

③

is bilinear (so $\langle \lambda\vec{x}, \lambda A\vec{x}\rangle = \lambda\langle \vec{x}, \lambda A\vec{x}\rangle$
$= \lambda^2\langle \vec{x}, A\vec{x}\rangle$).

$\blacksquare$

Now

$$2\langle \vec{x}, \vec{y}\rangle_A := \langle \vec{x}+\vec{y}, A(\vec{x}+\vec{y})\rangle - \langle \vec{x}, A\vec{x}\rangle - \langle \vec{y}, A\vec{y}\rangle$$

$$= \cancel{\langle \vec{x}, A\vec{x}\rangle} + \langle \vec{x}, A\vec{y}\rangle + \langle \vec{y}, A\vec{x}\rangle + \cancel{\langle \vec{y}, A\vec{y}\rangle}$$
$$- \cancel{\langle \vec{x}, A\vec{x}\rangle} - \cancel{\langle \vec{y}, A\vec{y}\rangle}$$

$$= \langle \vec{x}, A\vec{y}\rangle + \langle \vec{y}, A\vec{x}\rangle$$

$$= \langle \vec{x}, A\vec{y}\rangle + \langle A\vec{y}, \vec{x}\rangle$$

$$= 2\langle \vec{x}, A\vec{y}\rangle.$$

So

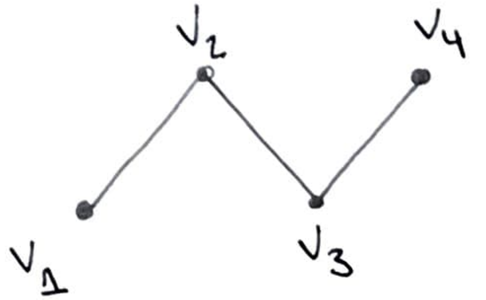
$$\langle \vec{x}, \vec{y}\rangle_A = \langle \vec{x}, A\vec{y}\rangle$$

and this is clearly bilinear in $\vec{x}$ and $\vec{y}$. $\square$

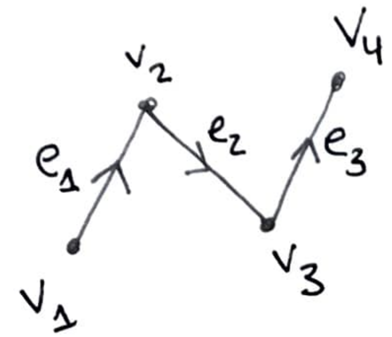
Now we are going to define a very important quadratic form for graphs.

Definition. If G is a graph with V vertices and E edges, then $\mathbb{R}^V$ is the space of <u>vertex</u> <u>weights</u> and $\mathbb{R}^E$ the space of <u>edge</u> <u>weights</u>.

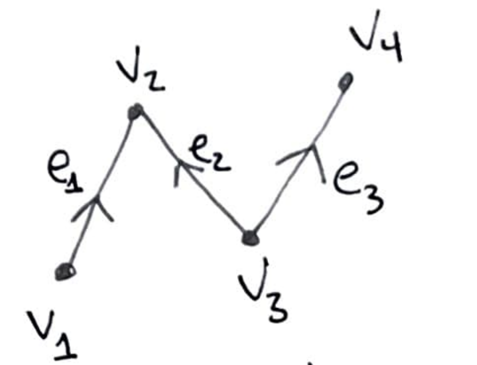
Definition. A graph G is <u>directed</u> or <u>oriented</u> if each edge has a <u>head</u> vertex and a <u>tail</u> vertex.



undirected



directed



directed

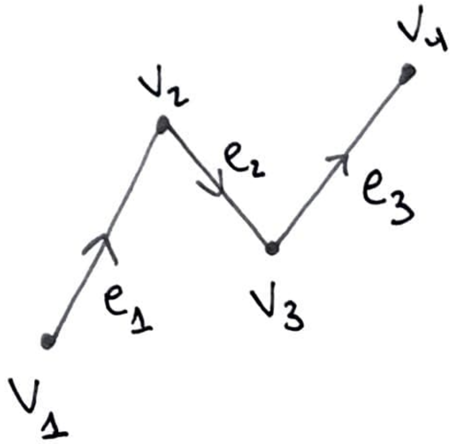
We denote orientation with an arrow tail $\bullet\!\longrightarrow\!\bullet$ head on each edge.

Definition. The <u>boundary map</u> $\partial : \mathbb{R}^E \to \mathbb{R}^V$ is the linear map defined by

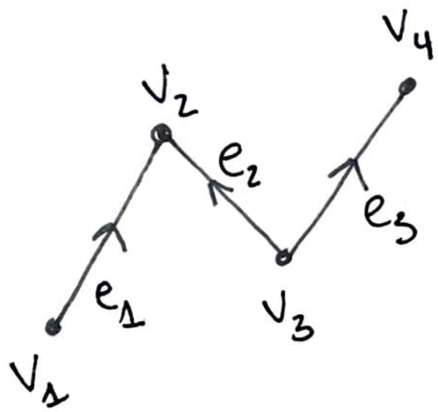
$$\partial(e_i) = \text{head}(e_i) - \text{tail}(e_i)$$

for an oriented graph G.

Examples.



$$\partial = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

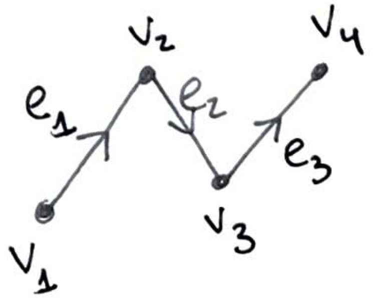


$$\partial = \begin{bmatrix} -1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Definition. The graph Laplacian L_G is the $V \times V$ symmetric matrix

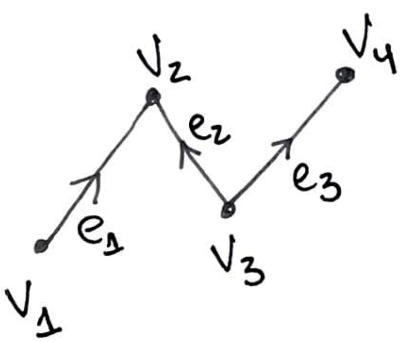
$$L_G := \partial \partial^T$$

Examples.



$$\partial\partial^T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$



$$\partial\partial^T = \begin{bmatrix} -1 & 0 & 0 \\ 1 & +1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

We notice two things by looking at this example.

1) Changing e_2 from $\underset{v_2}{\bullet} \longrightarrow \underset{v_3}{\bullet}$ to $\underset{v_2}{\bullet} \longleftarrow \underset{v_3}{\bullet}$ changed ∂, but not L_G.

2) $L_G = D_G - M_G$

We'll prove both in homework.

Proposition. As a quadratic form

$$Q_{L_G}(\vec{x}) = \sum_{i=1}^{|E|} (\vec{x}_{head(e_i)} - \vec{x}_{tail(e_i)})^2.$$

Proof.

$$Q_{L_G}(\vec{x}) = \langle \vec{x}, L_G \vec{x} \rangle$$

$$= \langle \vec{x}, \partial \partial^T \vec{x} \rangle$$

$$= \langle \partial^T \vec{x}, \partial^T \vec{x} \rangle$$

$$= \| \partial^T \vec{x} \|^2$$

Now $\partial^T : \mathbb{R}^V \to \mathbb{R}^E$ is defined by

$$\partial^T \delta(v_i) = \sum_{v_i = \text{head}(e_j)} \delta(e_j) - \sum_{v_i = \text{tail}(e_j)} \delta(e_j)$$

so

$$\partial^T \vec{x} = \sum_{j=1}^{E} (\vec{x}_{\text{head}(e_j)} - \vec{x}_{\text{tail}(e_j)}) \delta(e_j)$$

and $\| \partial^T \vec{x} \|^2 = \sum_{j=1}^{E} (\vec{x}_{\text{head}(e_j)} - \vec{x}_{\text{tail}(e_j)})^2$. $\square$

Definition. $\vec{x} \neq \vec{0}$ is an eigenvector of a matrix M with eigenvalue λ if $M\vec{x} = \lambda\vec{x}$.

Proposition. λ is an eigenvalue of M if and only $M - \lambda I$ is a singular matrix.

Proof. Recall that M is singular $\Leftrightarrow$ M is not invertible $\Leftrightarrow$ Ker M ~~is larger~~ contains a nonzero vector.

Thus λ is an eigenvalue $\Leftrightarrow \exists\, \vec{x} \neq \vec{0}$ so $M\vec{x} = \lambda\vec{x} \Leftrightarrow (M - \lambda I)\vec{x} = \vec{0} \Leftrightarrow$ $M - \lambda I$ is singular. $\square$

Definition. If M is an $n \times n$ matrix, the degree-n polynomial $p_M(t) = \det(M - tI)$ is called the <u>characteristic polynomial</u> of M.

Corollary (of last proposition) The eigenvalues of M are the roots of the characteristic polynomial.

Spectral Theorem. If M is an $n \times n$ real symmetric matrix, there are real $\lambda_1, \dots, \lambda_n$ and real, orthonormal vectors $\vec{\varphi}_1, \dots, \vec{\varphi}_n$ so that $M\vec{\varphi}_i = \lambda_i \vec{\varphi}_i$.

(We won't prove the spectral theorem, but notice how powerful it is! The fundamental theorem of algebra says that $P_M(t)$ has n <u>complex</u> roots, which may be repeated.)

Notes. Eigen<u>values</u> are uniquely determined, but eigen<u>vectors</u> are not. If

$$M\vec{\varphi}_i = \lambda_i \vec{\varphi}_i \quad \text{then} \quad M(-\vec{\varphi}_i) = \lambda_i(-\vec{\varphi}_i).$$

Further, if eigenvalues repeat, then $\lambda_i = \lambda_{i+1} = \dots = \lambda_{i+k}$ define a $k+1$ dimensional <u>eigenspace</u>, within which every vector is an eigenvector.

Definition. A matrix M is positive definite if all eigenvalues $\lambda_i > 0$, and positive semidefinite if all eigenvalues $\lambda_i \geq 0$.

Proposition. The graph Laplacian L_G is positive semidefinite.

Proof. Suppose $L_G \vec{\psi} = \lambda \vec{\psi}$. We know

$$\langle \vec{\psi}, \lambda \vec{\psi} \rangle = \langle \vec{\psi}, L_G \vec{\psi} \rangle$$

$$= \sum_{i=1}^{|E|} \left(\vec{\psi}_{head(e_i)} - \vec{\psi}_{tail(e_i)} \right)^2$$

$$\geq 0$$

But $\langle \vec{\psi}, \lambda \vec{\psi} \rangle = \lambda \| \vec{\psi} \|^2$. Thus $\lambda \geq 0$. $\square$

We will sort the eigenvalues of L_G from lowest to highest

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

Proposition. $\lambda_1 = 0$ for L_G.

(Homework)

We call λ_i with small i the "low frequency" and with large i the "high frequency" eigenvalues.