# Polygon Spaces and Grassmannians.

Idea of course.

Sometimes it's easiest to understand large theories if you have a ~~t~~ mastery of well-chosen examples. In this course, we're going to focus on polygon spaces as examples of several different theories - symplectic geometry, Riemannian geometry, and algebraic geometry.

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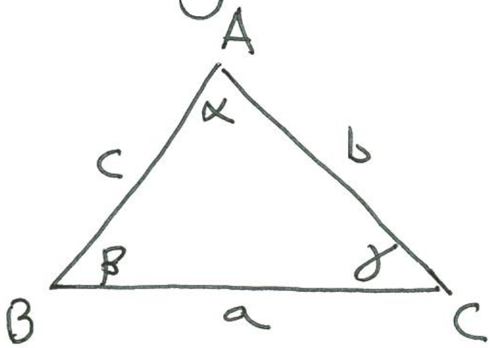
Definition. The space of plane polygons of length 2 with $n$ edges is denoted $Pol_2(n)$.

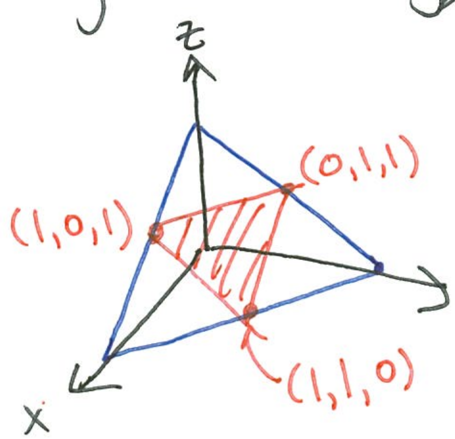
We want to describe $Pol_2(3)$ - triangle space.

Question. What fraction of triangle space consists of obtuse triangles?

Notation. We let the vertices sides and angles of the triangle be denoted




The triangles are ~~given~~ inside ~~by~~ $\{(a,b,c) \mid a+b+c=2\}$,




but we also have the triangle inequalities $a+b \geq c$, $b+c \geq a$, $c+a \geq b$

We change variables to

$$s_a = \frac{-a+b+c}{2}, \quad s_b = \frac{a-b+c}{2}, \quad s_c = \frac{a+b-c}{2}$$

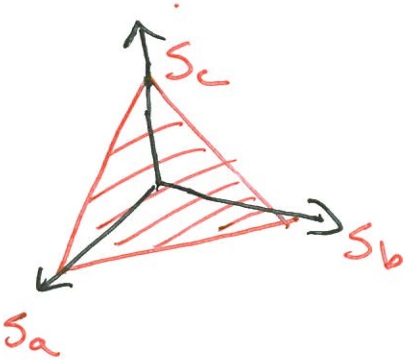
by the triangle inequalities, we know

$$s_a, s_b, s_c \geq 0.$$

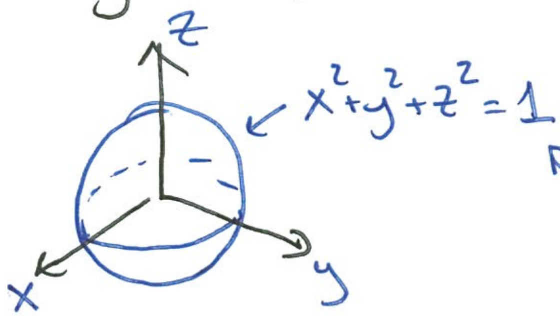
Further

$$s_a + s_b + s_c = \frac{a+b+c}{2} = 1,$$

so we can now write triangle space as the entire simplex.




Finally, we take square roots to let $x^2 = \cancel{\sqrt{\ }}s_a$, $y^2 = \cancel{\sqrt{\ }}s_b$, $z^2 = \cancel{\sqrt{\ }}c$. Then triangle space




8 fold cover of triangle space.

Proposition. Solving for a, b, and c,

$$a = 1-x^2, \quad b = 1-y^2, \quad c = 1-z^2.$$

Model 1. Triangle space is the positive orthant of the unit sphere.

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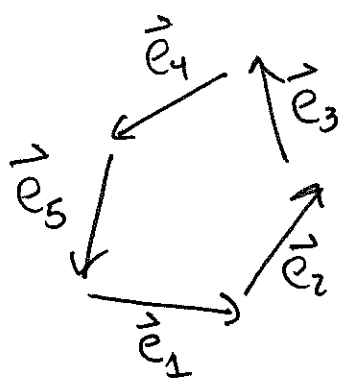
We are now going to reveal this as a (somewhat) more natural construction.




We think of polygons as edges instead of vertices, so triangle space is

$$\{(\vec{e}_1, \vec{e}_2, \vec{e}_3) \mid \vec{e}_1 + \vec{e}_2 + \vec{e}_3 = \vec{0}\}.$$

The edgelengths are $|\vec{e}_1| + |\vec{e}_2| + |\vec{e}_3| = 2$. As before, it probably gives us more symmetry to ~~square~~ & view the edgelengths as squares.

Let

$$e_1 = z_1^2, \quad \ldots, \quad e_3 = z_3^2$$

as complex numbers. If $e_1 + e_2 + e_3 = 0$, and $z_j = a_j + ib_j$, we have

$$\sum z_j^2 = \sum (a_j^2 - b_j^2) + (2a_j b_j)i$$

$$= 0 + 0i.$$

Thus

$$\sum a_j^2 = \sum b_j^2 \quad \text{and} \quad \sum a_j b_j = 0.$$

Proposition. If $z_j = a_j + b_j i$, then the polygon with edges $z_1^2, \ldots, z_n^2$ is closed as has length 2 $\Leftrightarrow$ the vectors $\vec{a} = (a_1, \ldots, a_n)$ and $\vec{b} = (b_1, \ldots, b_n)$ are orthonormal.

Now if we write

$$z_j = r_j e^{i\theta_j} = a_j + b_j i$$

we can see that rotating 

by $\theta$ rotates each $(a_j, b_j)$ by $\theta$ and rotates the ~~pol~~ edges $e_j = z_j^2$ of the polygon by $2\theta$.

We now need some ~~notation~~. definitions.

Definition. The Stiefel manifold $V_k(\mathbb{R}^n)$ is the space of orthonormal $k$-frames in $\mathbb{R}^n$. The Grassmann manifold $G_k(\mathbb{R}^n)$ is the space of $k$-dimensional linear subspaces of $\mathbb{R}^n$.

Thus

<u>triangle space is $2^3$-fold covered by $G_2(\mathbb{R}^3)$</u>

We can see

$$\overset{\text{planes}}{G_2(\mathbb{R}^3)} \cong \overset{\text{lines}}{G_1(\mathbb{R}^3)}$$

by taking normals of the planes. In fact, we can deduce the sidelengths in the "lines" model easily.

Model 2. A triangle is given by the line through the unit vector $\vec{v}$, as above.

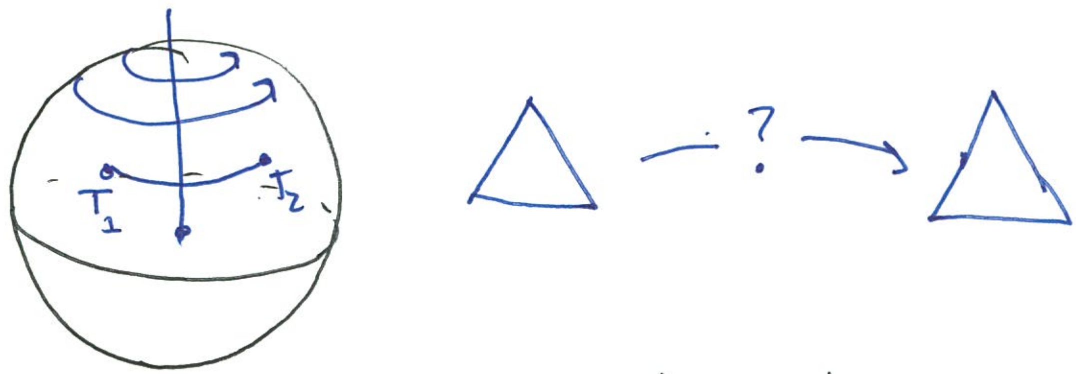
Proposition. Models 1 and 2 are the same.

If we complete $\vec{v}$ to an orthonormal $3\times3$ matrix $(\vec{a}, \vec{b}, \vec{v})$ then $a_i^2 + b_i^2 + v_i^2 = 1$. Thus $v_i^2 = 1 - (a_i^2 + b_i^2) = 1 - e_i$, and the triangle

$$\underset{\text{in model 2}}{\vec{v}} = \underset{\text{in model 1}}{(x, y, z)}$$

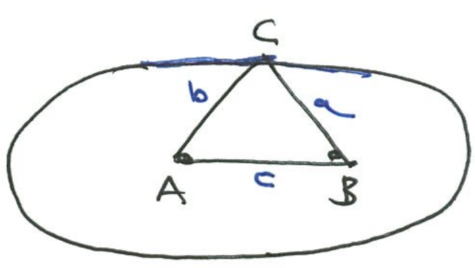
$\square$

This raises an interesting question:




What happens to a triangle when we rotate the sphere?

Proposition. The rotation of $(x,y,z)$ around the $z$-axis fixes sidelength $c = 1 - z^2$ and translates vertex $C$ around the ellipse with foci at $A$ and $B$ according to the parametrization $C(\theta) = (2\cos 2\theta, 2\sin 2\theta)$.



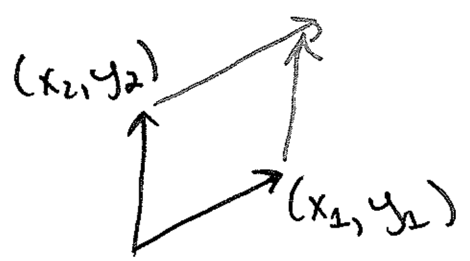

Proof. Since $c = 1 - z^2$ and $a + b + c = 2$, the vertex $C$ must move on the ellipse.

Claim. The parametrization $\gamma(t) = (m_1 \cos t, m_2 \sin t)$ sweeps out area at a constant rate.

We start by recalling the scalar cross product of vectors in the plane:

$$(x_1, y_1) \times (x_2, y_2) = x_1 y_2 - x_2 y_1$$

This is the area of the parallelogram




and obeys the "product rule"

$$\frac{d}{dt} v(t) \times w(t) = v'(t) \times w(t) + v(t) \times w'(t).$$

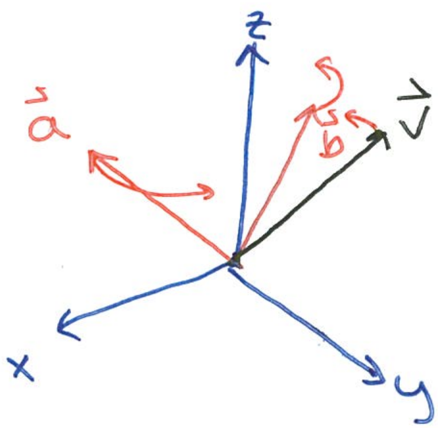
So, differentiating the area of the triangle $\gamma(t), \gamma(t+\epsilon)$ at $\epsilon = 0$, we see

$$\frac{d}{d\epsilon} \gamma(t) \times \gamma(t+\epsilon) = \gamma(t) \times \gamma'(t)$$

$$= (m_1 \cos t, m_2 \sin t) \times (-m_1 \sin t, m_2 \cos t)$$

$$= m_1 m_2 \cos^2 t + m_1 m_2 \sin^2 t = m_1 m_2.$$

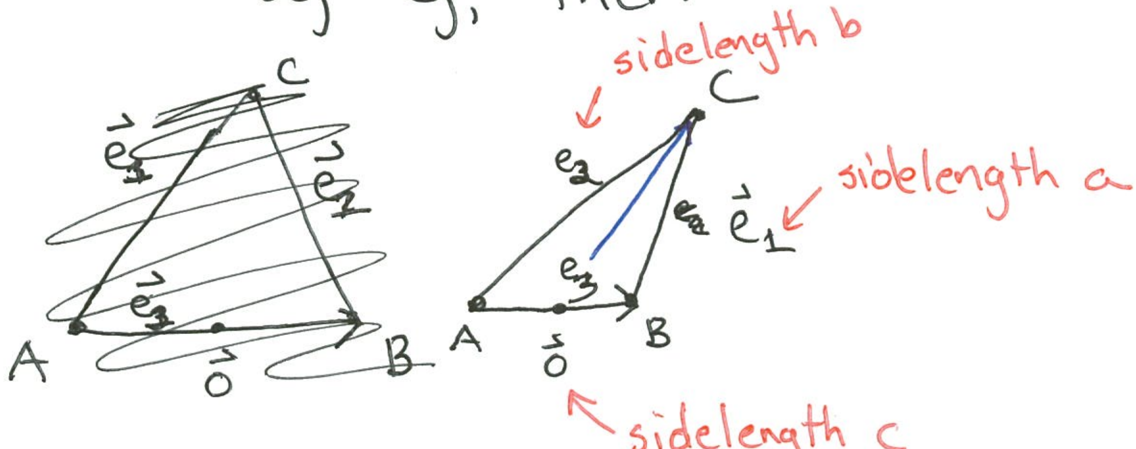
So it's enough to prove that rotating $\vec{v} = (x,y,z)$ around the z-axis causes vertex C to sweep out area at a constant rate.




Completing $\vec{v}$ to an orthonormal frame $(\vec{a}, \vec{b}, \vec{v})$ we see that $\vec{a}$ and $\vec{b}$ also rotate around the z-axis. If

$\omega_j = a_j + b_j i$ (sorry about the $\omega$, $z$ is taken now)

so that $\omega_j^2 = \vec{e}_j$, then



We can compute that C is at

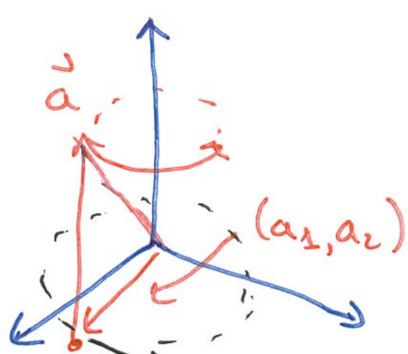
$$\frac{1}{2}e_3 + e_1 = \frac{1}{2}(-e_1 - e_2) + e_1$$

because $e_1 + e_2 + e_3 = 0$

$$= +\frac{1}{2}e_1 - \frac{1}{2}e_2$$

$$= \frac{1}{2}\left(\omega_1^2 - \omega_2^2\right)$$

Now $\omega_1 = a_1 + b_1 i$, and $\omega_2 = a_2 + b_2 i$ so to differentiate this, we need to know $\vec{a}'$ and $\vec{b}'$. The picture shows




$$(a_1, a_2)' = (-a_2, a_1)$$
$$(b_1, b_2)' = (-b_2, b_1)$$

$$(a_1, a_2)' = (a_1, a_2)^{\perp} = (-a_2, a_1) = (a_1, a_2, 0) \times \vec{z}.$$

Using these,

$$\omega_1' = a_1' + b_1' i = -a_2 - b_2 i = -\omega_2$$
$$\omega_2' = a_2' + b_2' i = a_1 + b_1 i = \omega_1$$

Thus as we rotate,

$$C \times C' = \frac{1}{2}(\omega_1^2 - \omega_2^2) \times \frac{d}{ds}\,\frac{1}{2}(\omega_1^2 - \omega_2^2)$$

$$= \frac{1}{4}(\omega_1^2 - \omega_2^2) \times (-2\omega_1\omega_2 - 2\omega_1\omega_2)$$

$$= -\cancel{\tfrac{2}{4}}(\omega_1^2 - \omega_2^2) \times \omega_1\omega_2$$

Now we need to connect x and complex numbers