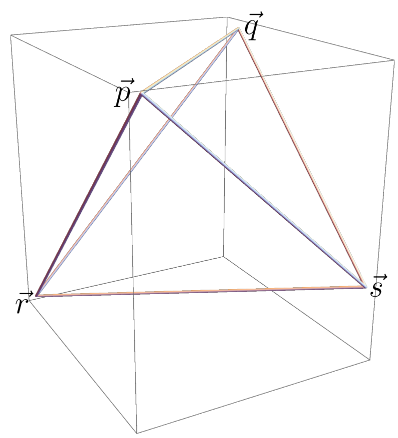
Math 4250/6250: The dot product, the point groups, and the regular solids.

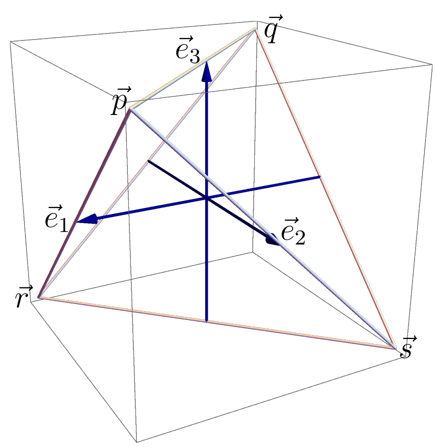
Definition 1 *A polyhedron $P \subset \mathbb{R}^3$ is a regular solid if every face is an identical regular polygon and the same number of faces meet at each vertex.*

You probably remember that there are only 5 regular solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, and I suspect that you can picture them. But here's a harder question: what are the coordinates of their vertices?

One way to generate coordinates is to use the point groups. For instance, any four (noncoplanar) points $\vec{p}, \vec{q}, \vec{r}, \vec{s} \in \mathbb{R}^3$ form a tetrahedron by taking the four triangular faces to be $\{\vec{q}, \vec{r}, \vec{s}\}$, $\{\vec{p}, \vec{r}, \vec{s}\}$, $\{\vec{p}, \vec{q}, \vec{s}\}$, and $\{\vec{p}, \vec{q}, \vec{r}\}$, as below.



Since three triangles meet at each vertex, the second condition of Definition 1 is met regardless of the positions of $\vec{p}, \vec{q}, \vec{r}, \vec{s}$. However, the triangular faces may not all be equilateral.



Here, $\vec{p} = (1,1,1)$, $\vec{q} = (-1,-1,1)$, $\vec{r} = (1,-1,-1)$ and $\vec{s} = (-1,1,-1)$. This is a special tetrahedron!

1. (10 points) We are now going to use the point group $\mathcal{G}$ to show that the tetrahedron above with $\vec{p} = (1,1,1)$, $\vec{q} = (-1,-1,1)$, $\vec{r} = (1,-1,-1)$ and $\vec{s} = (-1,1,-1)$ is a regular solid.

   (1) (5 points) Compute the matrix-vector products $A\vec{p}$, $A\vec{q}$, $A\vec{r}$, and $A\vec{s}$ and $B\vec{p}$, $B\vec{q}$, $B\vec{r}$, and $B\vec{s}$. Each of the 8 vectors you get will be equal to one of the original four vectors $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$. Record which one.

   Example.

$$A\vec{p} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{p}$$

   Note that this gives you a quick way to compute the matrix-vector product between any product of A's and B's and any of the four vectors $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$ without doing any additional matrix multiplication.

Solution: (Brute force solution)

$$A\vec{p} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{p}$$

and

$$A\vec{q} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \vec{r}$$

and

$$A\vec{r} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \vec{s}$$

and finally

$$A\vec{s} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \vec{q}.$$

(More clever solution) If we write out

$$A\vec{v} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_3 \\ v_1 \\ v_2 \end{pmatrix},$$

we see that A cyclically permutes the coordinates of any vector it is applied to.

Solution: Since all coordinates of $\vec{p} = (111)$ are the same, $A\vec{p} = \vec{p}$. Since the coordinates of $\vec{q}$, $\vec{r}$ and $\vec{s}$ are cyclic permutations of each other,

$$A\vec{q} = \vec{r}, \quad A\vec{r} = \vec{s}, \quad A\vec{s} = \vec{q}$$

(Brute force solution): For B, we compute

$$B\vec{p} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \vec{q}$$

and

$$B\vec{q} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{p}$$

and

$$B\vec{r} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \vec{s}$$

and finally

$$B\vec{s} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \vec{r}.$$

(More clever solution): You still have to compute $B\vec{p} = \vec{q}$ and $B\vec{r} = \vec{s}$. But because $B^2 = I$, you can multiply by B on the left on both sides to get $(BB)\vec{p} = \vec{p} = B\vec{q}$ and $(BB)\vec{r} = \vec{r} = B\vec{s}$.

Grading notes: Accept either the brute force or cleverer solutions for full credit, but praise the clever solutions somehow by writing something encouraging.

Rubric:

Computational error	-2 (each, minimum 1 pt)
Messy	-1

(2) (5 points) We will identify each edge of the tetrahedron by the set of vertices that it connects. For example, the edge joining $\vec{p}$ and $\vec{q}$ will be referred to as $\{\vec{p}, \vec{q}\}$. Since sets are not ordered, $\{\vec{p}, \vec{q}\} = \{\vec{q}, \vec{p}\}$.

Since our isometries in $\mathcal{G}$ (that is, our products of A's and B's) are linear maps, if a matrix $M \in \mathcal{G}$ takes the endpoints of one edge to the endpoints of another[1], such as in the example:

$$M\{\vec{p}, \vec{q}\} = \{M\vec{p}, M\vec{q}\} = \{\vec{r}, \vec{s}\}$$

then M takes all the points on the edge joining $\vec{p}$ to $\vec{q}$ to corresponding points on the edge joining $\vec{r}$ and $\vec{s}$. In particular, since isometries preserve distances between vectors, the existence of such an $M \in \mathcal{G}$ would prove that

$$\|\vec{p} - \vec{q}\| = \|M\vec{p} - M\vec{q}\| = \|\vec{r} - \vec{s}\|.$$

Use the results of the last problem to find six matrices in $\mathcal{G}$ which take the edge $\{\vec{p}, \vec{q}\}$ to each of the six edges of the tetrahedron: $\{\vec{p}, \vec{q}\}$,$\{\vec{p}, \vec{r}\}$, $\{\vec{p}, \vec{s}\}$, $\{\vec{q}, \vec{r}\}$, $\{\vec{q}, \vec{s}\}$, and $\{\vec{r}, \vec{s}\}$. Note that this proves that all four faces are equilateral triangles.

Solution: (Expected solution): The easy ones are

$$A\{\vec{p}, \vec{q}\} = \{\vec{p}, \vec{r}\}$$
$$AA\{\vec{p}, \vec{q}\} = \{\vec{p}, \vec{s}\}.$$

If we apply B, we get

$$BA\{\vec{p}, \vec{q}\} = \{\vec{q}, \vec{s}\}$$
$$BAA\{\vec{p}, \vec{q}\} = \{\vec{q}, \vec{r}\}.$$

The last one takes a little experimentation, but it turns out that

$$ABAA\{\vec{p}, \vec{q}\} = \{\vec{r}, \vec{s}\}.$$

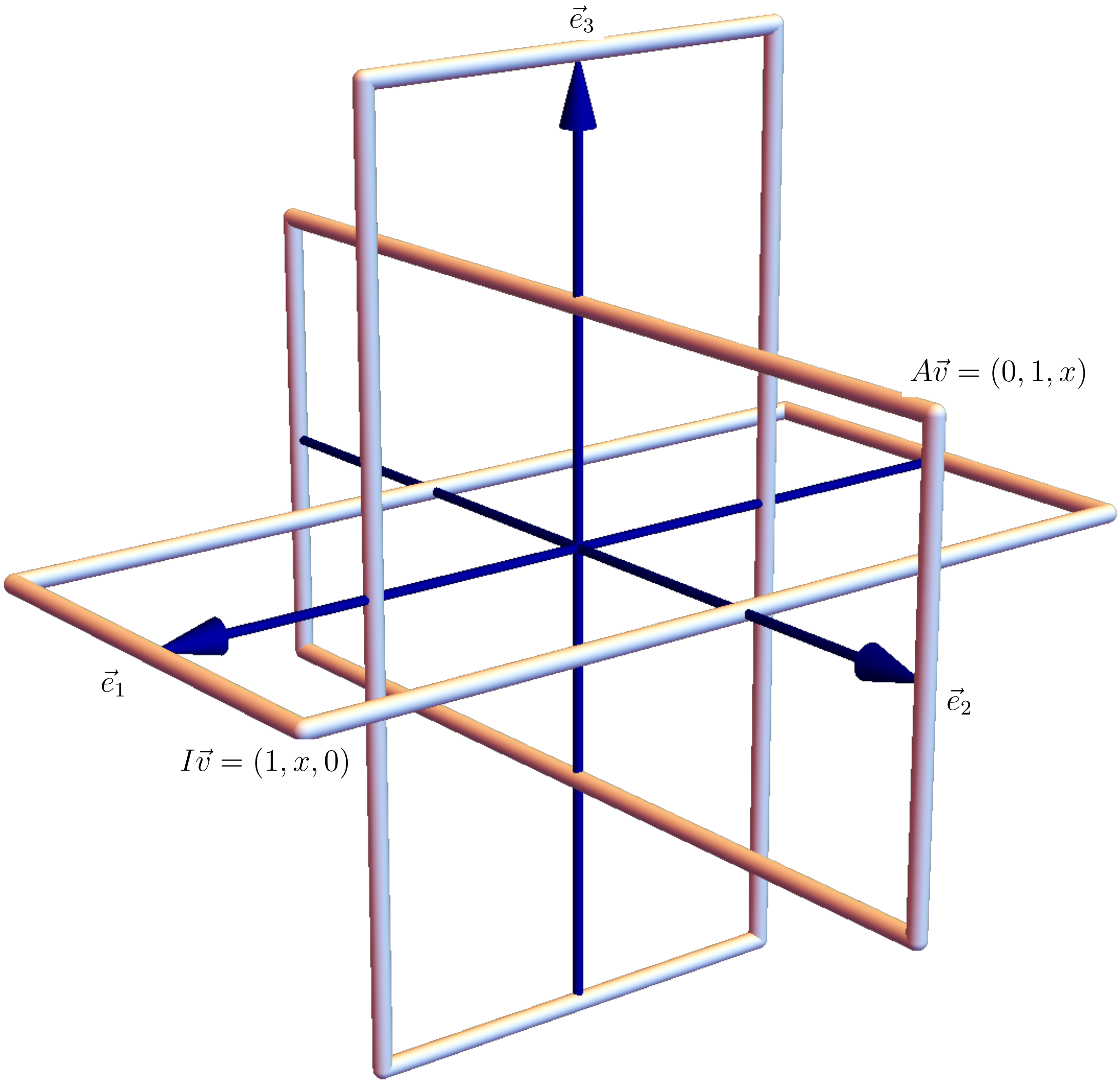
Grading notes: It's likely that some students will just start applying matrices from their previous list to the actual coordinates of the vertices and looking for matches experimentally. This isn't exactly *wrong*, but it's certainly to be discouraged. Don't deduct points (if they get workable answers eventually), but write a comment about "Use the last question".

Rubric:

Computational error	-2 (each, minimum 1 pt)
Messy	-1

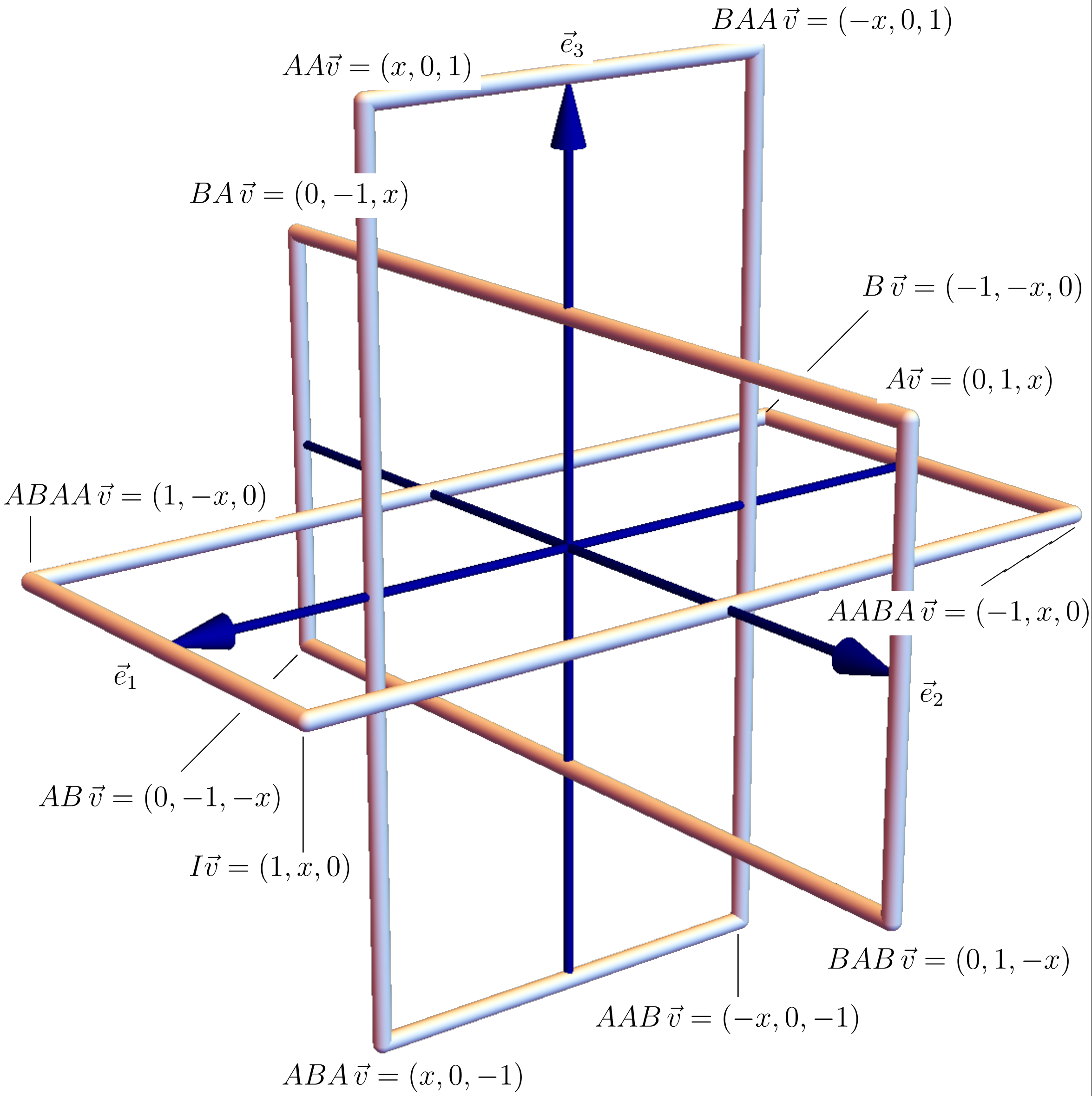
[1] In either order!

2. (10 points) Starting with any $\vec{v} = (1, x, 0)$ (assume $x < 1$), we can generate 12 vectors $\vec{v}_1, \ldots, \vec{v}_{12}$ by applying the 12 matrices in $\mathcal{G}$ to $\vec{v}$. We have can group these into the vertices of 3 rectangles in the $\vec{e}_1 - \vec{e}_2$, $\vec{e}_2 - \vec{e}_3$ and $\vec{e}_3 - \vec{e}_1$ planes as below.



Label each vertex above with its coordinates **and the corresponding matrix in** $\mathcal{G}$ (written as a product of A's and B's). We have labeled $I\vec{v}$ and $A\vec{v}$ above to help you get started.

Solution:



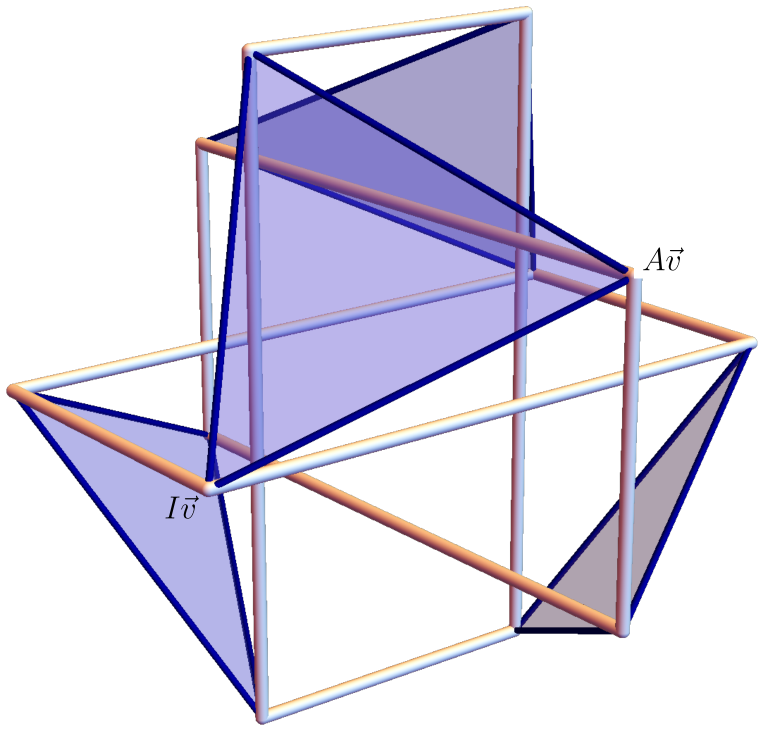
Grading notes: The students are likely to get the coordinates right, but the group elements wrong. They may even fail to write down the group elements at all! These are significant errors (not just oversights), because the point of the problem is for them to *connect the matrices in the point group* with the picture).

Rubric:

Point has wrong coords or group element	-2 (each, minimum 1 pt)
Doesn't write matrices or group elements	-6
Writes matrices instead of group elements	-4
Messy	-2

3. (15 points) As we did with the tetrahedron, we're now going to use the point group to show that certain distances between our 12 points are the same and we're going to connect this group to a different Platonic solid!

   (1) (5 points) The edge $\{\vec{v}, A\vec{v}\}$ is one of 12 edges marked in blue on the picture below. Use the results of Question 2 to describe each of these edges in the form $\{C\vec{v}, D\vec{v}\}$ where C and D are products of A's and B's.



There is plenty of space to write computations below and on the next page, but it might be easier for you to write in the coordinates on the picture above.

Solution: The blue edges are

$$\begin{array}{ccc} \{I, A\} & \{A, AA\} & \{AA, I\} \\ \{B, BA\} & \{BA, BAA\} & \{BAA, B\} \\ \{ABA, ABAA\} & \{ABAA, AB\} & \{AB, ABA\} \\ \{AAB, AABA\} & \{AABA, BAB\} & \{BAB, AAB\} \end{array}$$

Grading notes: This is essentially computational, so you're just seeing if they do the computation accurately. It's ok to label the edges on the diagram rather than writing out a list.

Rubric:

Computational error	-2 (each, minimum 1 pt)
Messy	-1

Solution:

(2) (5 points) Prove that all of these edges have the same length by finding isometries in $\mathcal{G}$ which take $\{I\vec{v}, A\vec{v}\}$ to each of the other blue edges. This proves that the blue triangles are equilateral. Hint: You'll eventually need to use the relations between products of A and B that you developed from $(AB)^3 = I$ in the last homework.

Solution: The first two edges are easy:

$$A\{I, A\} = \{A, AA\}$$
$$AA\{I, A\} = \{AA, AAA\} = \{A, I\}$$

Once we can take $\{I, A\}$ to $\{A, AA\}$ and $\{AA, I\}$, we only need to find isometries which take this group of three edges onto other groups of three, such as:

$$B\{I, A\} = \{B, BA\}$$
$$B\{A, AA\} = \{BA, BAA\}$$
$$B\{AA, I\} = \{BAA, B\}.$$

We can in addition use the relation $AAA = I$:

$$ABA\{I, A\} = \{ABA, ABAA\}$$
$$ABA\{A, AA\} = \{ABAA, ABAAA\} = \{ABAA, AB\}$$
$$ABA\{AA, I\} = \{ABAAA, ABA\} = \{AB, ABA\}.$$

The last group is harder and we need the relations from the last homework:

$$AAB = BABA \qquad BAA = ABAB \qquad ABA = BAAB$$
$$ABAA = BAABA \quad BAB = AABAA \quad AABA = BABAA$$

Using these, we can show

$$AAB\{I, A\} = \{AAB, AABA\}$$
$$AAB\{A, AA\} = \{AABA, AABAA\} = \{AABA, BAB\}$$
$$AAB\{AA, I\} = \{AABAA, AAB\} = \{BAB, AAB\}.$$

Grading notes: The biggest danger here will be that the students work out the length from the coordinates of the vertices (instead of actually finding the group elements) and then conclude that the lengths are all the same without actually doing the question.

Rubric:

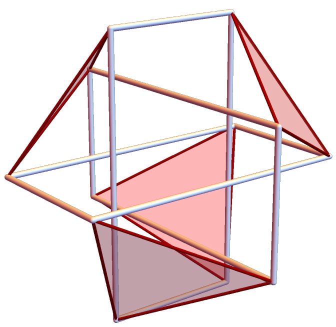
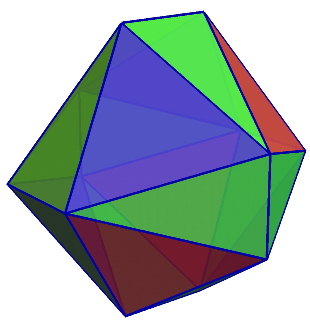
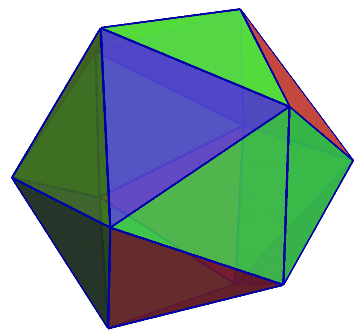
Computes lengths from coords	-3 (and note)
Computational error	-2 (each, minimum 1 pt)
Messy	-1

(3) (5 points) You don't have to match isometries with edges explicitly again, but the 12 isometries in $\mathcal{G}$ map the edge $\{\vec{v}, BAB\vec{v}\}$ to the 12 edges in red below left. The red edges all have the same length and the red triangles are equilateral. So our construction yields a one-parameter family of solids which are $\mathcal{G}$-symmetric, depending on the x in $\vec{v} = (1, x, 0)$.

Each has 12 vertices and 20 triangular faces, with 5 triangles meeting at every vertex. However, while the 4 red triangles and the 4 blue triangles are always equilateral, the 12 green triangles are only isoceles. An example is shown below center.

Solve for the length of the red edges $r(x)$ and the blue edges $b(x)$ to prove that $b(x) = r(x)$. Then set $b(x) = r(x) = 2x$ (the length of the short side of the rectangles) to find the x which makes the green triangles equilateral and the entire figure an icosahedron, as shown below right.

Hint: The value x should be familiar ... what is it?

Solution: Since $\vec{v} = (1, x, 0)$ and $A\vec{v} = (0, 1, x)$, the length

$$b(x) = \sqrt{1^2 + (1-x)^2 + x^2} = \sqrt{2 - 2x + 2x^2} = \sqrt{2}\,\sqrt{x^2 - x + 1}.$$

Similarly, since $BAB\vec{v} = (0, 1, -x)$, the length $r(x) = \sqrt{2}\,\sqrt{x^2 - x + 1}$. Solving

$$\sqrt{2}\sqrt{x^2 - x + 1} = 2x \quad \text{yields} \quad x = \frac{-1 + \sqrt{5}}{2}$$

which is the golden ratio!

Grading notes: Praise students who recognize the golden ratio, but don't deduct points if they don't get it.

Rubric:

Algebra error	-2 (each, minimum 1 pt)
Messy	-1

Solution: