Math 4250/6250: A tale of two matrices.

In the last homework, you proved that if A and B are orthogonal $n \times n$ matrices, then AB is an orthogonal matrix and A^{-1} is an orthogonal matrix. By doing so, you proved that the collection of orthogonal $n \times n$ matrices forms a "group". (Here, *group* is used in the algebraic sense of MATH 4000/4010, not just colloquially to mean a "collection" or "set" of matrices.) This group is called $O(n)$.

Definition 1 *A set of matrices* $\mathcal{G} \subset O(n)$ *forms a* subgroup *of* $O(n)$ *if two properties hold. First, for any* $A, B \in \mathcal{G}$*, we have* $AB \in \mathcal{G}$*. Second, if* $A \in \mathcal{G}$*, then* $A^{-1} \in \mathcal{G}$*.*

It's a fascinating fact about our 3-dimensional world that there are only a short list of finite subgroups of $O(3)$. They are generally called the "point groups". This fact controls aspects of chemistry, crystallography, and mathematics. We are now going to explore one of the point groups. Consider the two matrices from the video:

$$\underbrace{A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{\text{permute } \vec{e}_1 \to \vec{e}_2 \to \vec{e}_3} \quad \text{and} \quad \underbrace{B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{rotate by } \pi \text{ around } \vec{e}_3}.$$

1. (10 points) We are now going to prove two things about A and B.
   (1) (5 points) Verify for yourself that A, $B \in O(3)$. (Note: This is in the video, but I'd like you to check it by hand.)

(2) (5 points) Show by direct matrix multiplication that $B^2 = A^3 = (AB)^3 = I$. (It's ok to use a computer; just include screenshots.)

2. (10 points) *Use the relations* $B^2 = A^3 = (AB)^3 = I$ to prove there are only 12 unique products of A's and B's, including I. These 12 matrices form a point group we'll call $\mathcal{G}$. Write out each matrix (you'll need them later, so do this carefully). While it's ok to use the brute force method from the video, it's much better to try to use the insightful method!

4