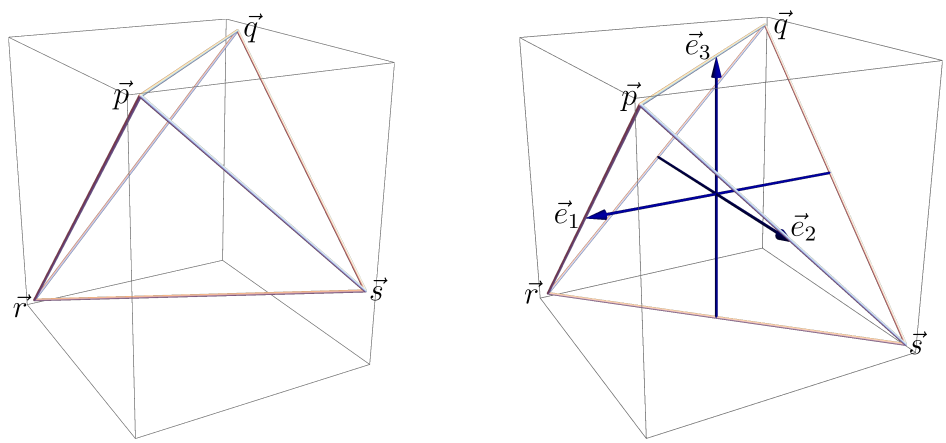
Math 4250/6250: The dot product, the point groups, and the regular solids.

Definition 1 *A polyhedron $P \subset \mathbb{R}^3$ is a regular solid if every face is an identical regular polygon and the same number of faces meet at each vertex.*

You probably remember that there are only 5 regular solids: the tetrahedron, cube, octahedron, dodecahedron, and icosahedron, and I suspect that you can picture them. But here's a harder question: what are the coordinates of their vertices? Surely they should have some beautiful structure of their own! One way to generate coordinates is to use the point groups.[1] For instance, any four (noncoplanar) points $\vec{p}, \vec{q}, \vec{r}, \vec{s} \in \mathbb{R}^3$ form a tetrahedron by taking the four triangular faces to be $\{\vec{q}, \vec{r}, \vec{s}\}$, $\{\vec{p}, \vec{r}, \vec{s}\}$, $\{\vec{p}, \vec{q}, \vec{s}\}$, and $\{\vec{p}, \vec{q}, \vec{r}\}$, as below.



Since three triangles meet at each vertex, the second condition of Definition 1 is met regardless of the positions of $\vec{p}, \vec{q}, \vec{r}, \vec{s}$. However, the triangular faces may not all be equilateral. However, if we choose the four points to be opposite pairs of corners of the top and bottom face of a cube (as above right), then $\vec{p} = (1, 1, 1)$, $\vec{q} = (-1, -1, 1)$, $\vec{r} = (1, -1, -1)$ and $\vec{s} = (-1, 1, -1)$. This is certainly a special tetrahedron, and you will show over the course of this homework that it is a regular tetrahedron.

[1] This approach is from Section 3.77 of Coxeter's beautiful book *Regular Polytopes*. If you want to know more about this, the relevant chapter is posted on our class webpage.

1. (10 points) We are now going to use the point group $\mathcal{G}$ from the last homework to show that the tetrahedron above with $\vec{p} = (1, 1, 1)$, $\vec{q} = (-1, -1, 1)$, $\vec{r} = (1, -1, -1)$ and $\vec{s} = (-1, 1, -1)$ is a regular solid. Recall that $\mathcal{G}$ was generated by two matrices:

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and that the group $\mathcal{G}$ contained 12 different products of A's and B' (including the identity matrix).

(1) (5 points) Compute the matrix-vector products $A\vec{p}$, $A\vec{q}$, $A\vec{r}$, and $A\vec{s}$ and $B\vec{p}$, $B\vec{q}$, $B\vec{r}$, and $B\vec{s}$. Each of the 8 vectors you get will be equal to one of the original four vectors $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$. Record which one.

Example.

$$A\vec{p} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \vec{p}$$

Note that this gives you a quick way to compute the matrix-vector product between any product of A's and B's and any of the four vectors $\vec{p}$, $\vec{q}$, $\vec{r}$, and $\vec{s}$ without doing any additional matrix multiplication.

(2) (5 points) We will identify each edge of the tetrahedron by the set of vertices that it connects. For example, the edge joining $\vec{p}$ and $\vec{q}$ will be referred to as $\{\vec{p}, \vec{q}\}$. Since sets are not ordered, $\{\vec{p}, \vec{q}\} = \{\vec{q}, \vec{p}\}$.

Since our isometries in $\mathcal{G}$ (that is, our products of A's and B's) are linear maps, if a matrix $M \in \mathcal{G}$ takes the endpoints of one edge to the endpoints of another[2], such as in the example:

$$M\{\vec{p}, \vec{q}\} = \{M\vec{p}, M\vec{q}\} = \{\vec{r}, \vec{s}\}$$

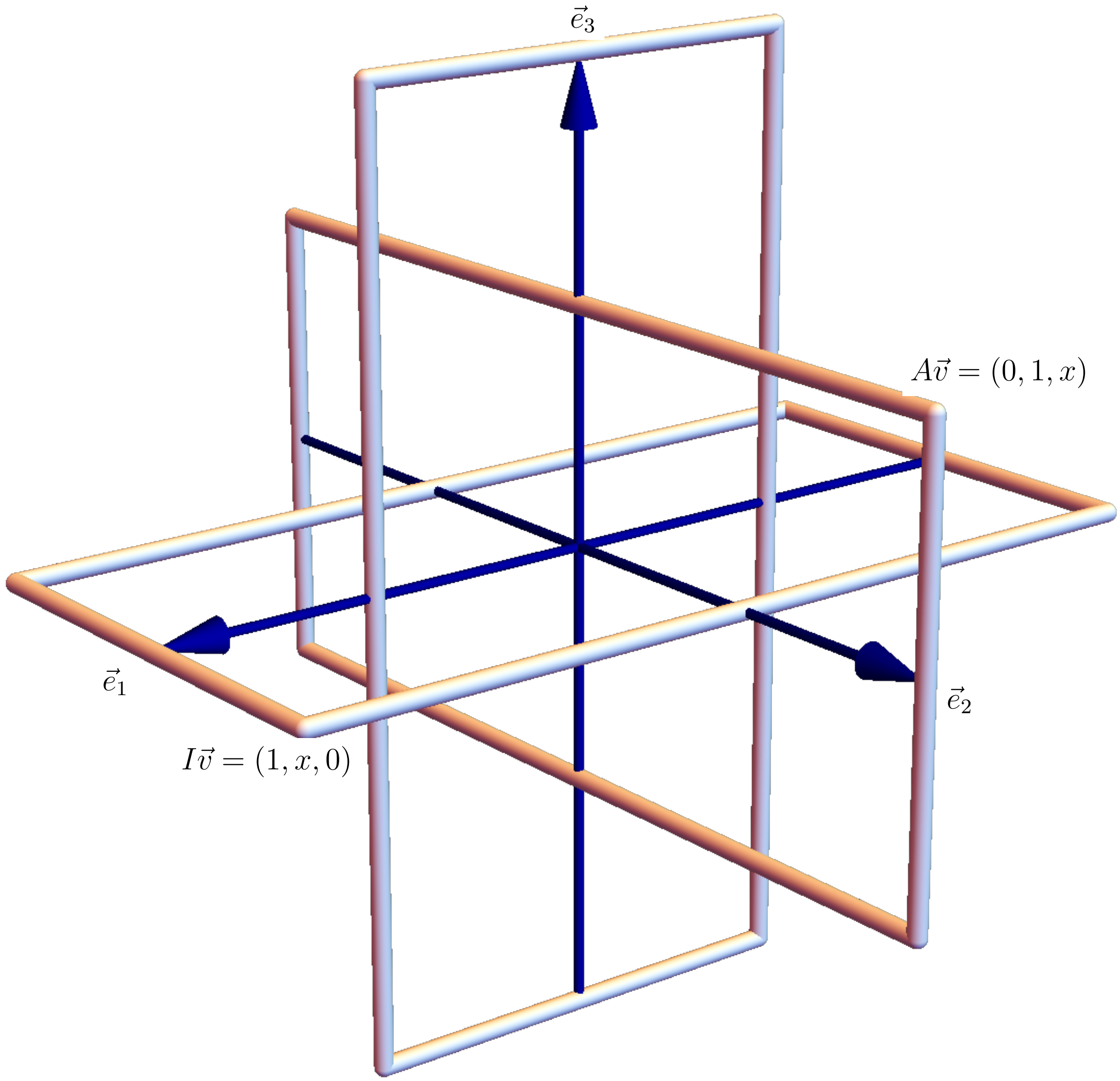
then M takes all the points on the edge joining $\vec{p}$ to $\vec{q}$ to corresponding points on the edge joining $\vec{r}$ and $\vec{s}$. In particular, since isometries preserve distances between vectors, the existence of such an $M \in \mathcal{G}$ would prove that

$$\|\vec{p} - \vec{q}\| = \|M\vec{p} - M\vec{q}\| = \|\vec{r} - \vec{s}\|.$$

Use the results of the last problem to find six matrices in $\mathcal{G}$ which take the edge $\{\vec{p}, \vec{q}\}$ to each of the six edges of the tetrahedron: $\{\vec{p}, \vec{q}\}$,$\{\vec{p}, \vec{r}\}$, $\{\vec{p}, \vec{s}\}$, $\{\vec{q}, \vec{r}\}$, $\{\vec{q}, \vec{s}\}$, and $\{\vec{r}, \vec{s}\}$. Note that this proves that all four faces are equilateral triangles.

[2] In either order!

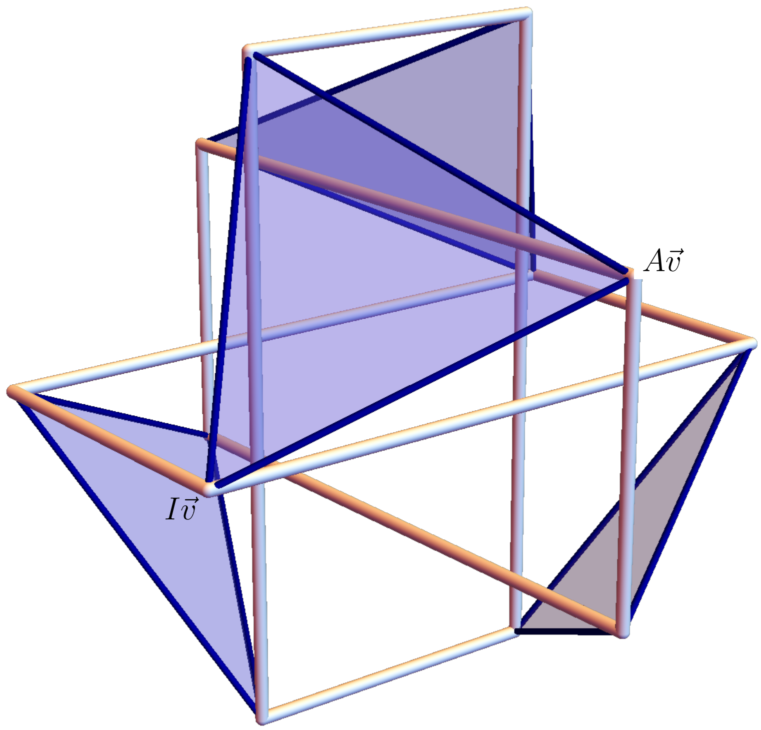
2. (10 points) Starting with any $\vec{v} = (1, x, 0)$ (assume $x < 1$), we can generate 12 vectors $\vec{v}_1, \ldots, \vec{v}_{12}$ by applying the 12 matrices in $\mathcal{G}$ to $\vec{v}$. We have grouped these into the vertices of 3 rectangles in the $\vec{e}_1 - \vec{e}_2$, $\vec{e}_2 - \vec{e}_3$ and $\vec{e}_3 - \vec{e}_1$ planes as below.



Label each vertex above with its coordinates **and the corresponding matrix in** $\mathcal{G}$ (written as a product of A's and B's). We have labeled $I\vec{v}$ and $A\vec{v}$ above to help you get started.

3. (10 points) As we did with the tetrahedron, we're now going to use the point group to show that certain distances between our 12 points are the same and we're going to connect this group to a different Platonic solid!

   (1) (5 points) The edge $\{\vec{v}\, A\vec{v}\}$ is one of 12 edges marked in blue on the picture below. Use the results of Question 2 to describe each of these edges in the form $\{C\vec{v}\, D\vec{v}\}$ where C and D are products of A's and B's.



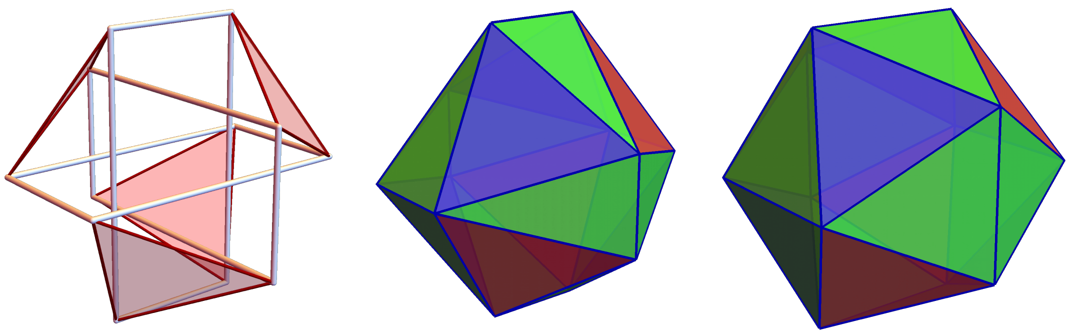
(2) (5 points) Prove that all of these edges have the same length by finding isometries in $\mathcal{G}$ which take $\{I\vec{v}\,A\vec{v}\}$ to each of these edges. This proves that the blue triangles are equilateral. Hint: You'll eventually need to use the relations between products of A and B that you developed from $(AB)^3 = I$ in the previous minihomework.

(3) (5 points) You don't have to match isometries with edges explicitly again, but the 12 isometries in $\mathcal{G}$ map the edge $\{\vec{v}, BAB\vec{v}\}$ to the 12 edges in red below left. The red edges all have the same length and the red triangles are equilateral. So our construction yields a one-parameter family of solids[3] which are $\mathcal{G}$-symmetric, depending on the x in $\vec{v} = (1, x, 0)$.

Each has 12 vertices and 20 triangular faces, with 5 triangles meeting at every vertex. However, while the 4 red triangles and the 4 blue triangles are always equilateral, the 12 green triangles are only isoceles. An example is shown below center.

Solve for the length of the red edges $r(x)$ and the blue edges $b(x)$ to prove that $b(x) = r(x)$. Then set $b(x) = r(x) = 2x$ (the length of the short side of the rectangles) to find the x which makes the green triangles equilateral and the entire figure an icosahedron, as shown below right.

Hint: The value x should be familiar ... what is it?



[3]This family was called a *jitterbug* family by Buckminster Fuller, as mentioned in the video. For more on this, see Verheyn's *The complete set of Jitterbug transformers and the analysis of their motion*, which is posted on the course webpage.