Parametrized curves, examples and constructions

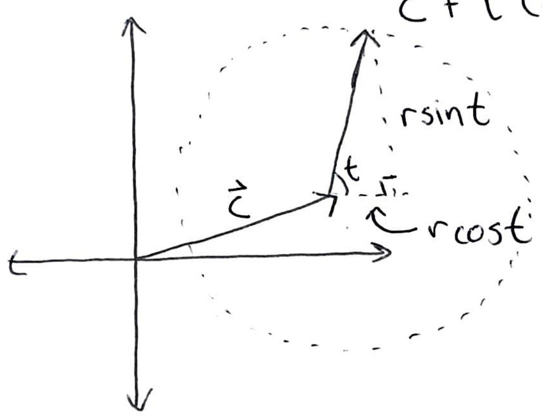
Recall that a parametrized curve is a map $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$. We will now study some example curves.

Example. The circle of radius r with center $\vec{c} = (c_1, c_2)$ in $\mathbb{R}^2$ is described implicitly bey

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$

We can parametrize this curve by

$$\vec{c} + r(\cos t, \sin t) = \vec{\alpha}(t)$$



Notice that

$$\vec{\alpha}(t) = (c_1 + r\cos t,\ c_2 + r\sin t)$$

obeys

$$(\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 =$$

$$= (r\cos t)^2 + (r\sin t)^2$$

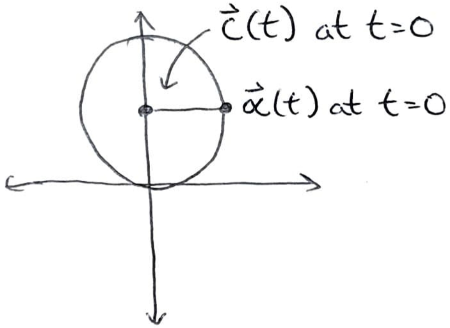
$$= r^2(\cos^2 t + \sin^2 t) = r^2,$$

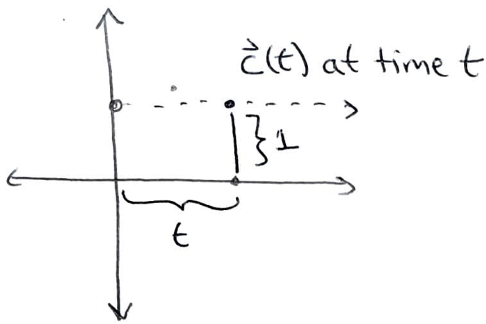
but there is more information in the ~~para~~ parametrization $\vec{\alpha}(t)$ because it tells us <u>when</u> each point on the circle is reached.

Example 2. $\vec{\alpha}(t) = (c_1 + r\cos(t^2), c_2 + r\sin(t^2))$ also parametrizes the circle of radius r and center $\vec{c} = (c_1, c_2)$.

We can make some beautiful curves by combining sines and cosines.

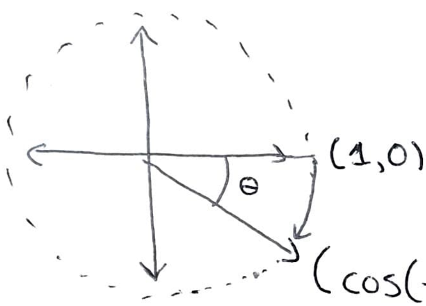
Example. A unit circle starts with center at $(0,1)$ and rolls along the pos. x axis. Parametrize the path of a point starting at $(1,1)$.





If the center of the circle is given by $\vec{c}(t)$, we can assume that the circle is rolling to the right at unit speed, so $\vec{c}(t) = (t,1)$.

However, if ~~th~~ a unit circle has rolled t units forward, it has turned by an angle of t radians... in the <u>clockwise</u> direction.



$(\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$

This rotation carries the point at $(1,0)$ ~~to~~ ~~the~~ (relative to the center) to the point at $(\cos\theta, -\sin\theta)$ (relative to the center).

Adding these together:

$$\vec{\alpha}(t) = (t + \cos t, 1 - \sin t)$$

We will work through a more elaborate ~~ex~~ example of this type of motion in homework when we describe the square-wheeled car.

We often describe curves with a differential equation, so let's remember how to solve (easy) ODEs.

If $u'(t) = F(u(t))$, then

$$\frac{u'(t)}{F(u(t))} = 1$$

integrate w.r.t. t

$$\int \frac{u'(t)}{F(u(t))}\,dt = \int 1\,dt$$

$du = u'(t)\,dt$

$$\int \frac{1}{F(u)}\,du = \int 1\,dt$$

so $\int 1/F(u)\,du = t + C$.

If we can do the integral on the left to get some

$$G(u) = \int \frac{1}{F(u)}\,du$$

then we get an equation

$$G(u) = t$$

which we can try to solve for $u(t)$.

Example. $u'(t) = u(t)^2$.

$$\frac{u'(t)}{u(t)^2} = 1 \quad \Rightarrow \quad \int \frac{1}{u(t)^2}\, u'(t)\,dt = \int 1\,dt$$

$$\Rightarrow \int \frac{1}{u^2}\,du = t + C \quad \Rightarrow \quad -\frac{1}{u} = t + C$$

$$\Rightarrow u = -\frac{1}{t+C}$$

So $u(t) = -\dfrac{1}{t+C}$, and indeed $u'(t) = \dfrac{1}{(t+c)^2} = u(t)^2$.