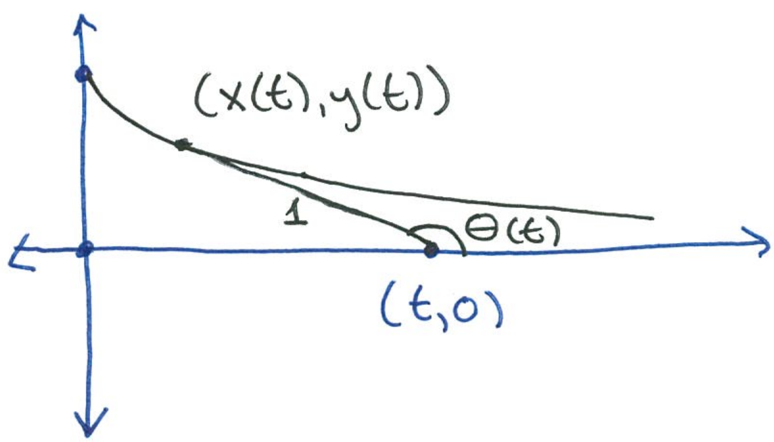
The tractrix.

A mass is located at $(0,1)$ and pulled by a linkage of fixed length 1 moving along the x-axis at speed 1.

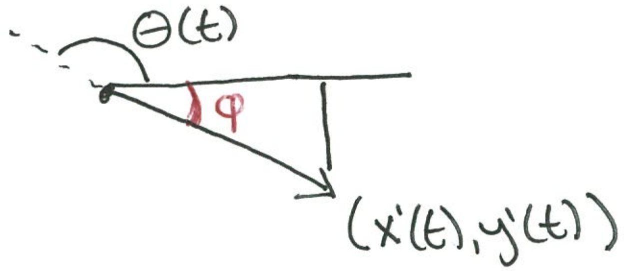


We know

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

because of the length-1 constraint.

Less obviously, the linkage is tangent to the curve, so we know that we have a triangle



Since

$$\tan\varphi = \frac{-y'(t)}{x'(t)}$$

recall $y'(t)$ is negative!

and $\varphi = \pi - \theta$, the supplementary angle formula for tan tells us that

$$\tan\theta = \frac{y'(t)}{x'(t)} = \frac{\cos\theta\ \theta'(t)}{1 - \sin\theta\ \theta'(t)}$$

We can solve this formula for $\theta'(t)$.

$$\tan\theta\,(1 - \sin\theta\,\theta') = \cos\theta\,\theta'$$

$$\tan\theta - \tan\theta\sin\theta\,\theta' = \cos\theta\,\theta'$$

$$\tan\theta = \left(\cos\theta + \frac{\sin^2\theta}{\cos\theta}\right)\theta'$$

→ multiplying through by cos.

$$\sin\theta = \theta'$$

We can solve this by separation of variables: $\sin\theta = \frac{d\theta}{dt}$, so

$$\int \frac{1}{\sin\theta}\,d\theta = \int 1\,dt$$

~~and~~ or

$$\int \csc\theta\,d\theta = -\ln(\csc\theta + \cot\theta) + C$$

$$= t$$

for some constant c.

at $t=0$, we have $\theta = \pi/2$, so

$$\csc \pi/2 = 1, \quad \cot \pi/2 = \frac{0}{1} = 0$$

and $-\ln(\csc \pi/2 + \cot \pi/2) = -\ln 1 = 0$.

This means $c=0$. So

$$t = -\ln\left(\csc\theta + \cot\theta\right).$$

We want to solve this for θ.

Now

$$\csc\theta + \cot\theta = \frac{1+\cos\theta}{\sin\theta}$$

We Know

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}$$

$$\sin\theta = 2\cos\theta/2 \, \sin\theta/2$$

so

$$\frac{1+\cos\theta}{\sin\theta} = \frac{2\cos^2 \theta/2}{2\cos \theta/2 \, \sin\theta/2}$$

$$= \cancel{\tan}\ \cot \theta/2$$

(who said trig was useless!?) and

$$t = +\ln \tan \theta/2$$

we switched from cot to tan, killing the minus sign.

so the tractrix is parametrized by θ if we substitute this back into

$$x(t) = t + \cos\theta$$
$$y(t) = \sin\theta$$

to get

$$x(\theta) = \cos\theta + \ln \tan \theta/2$$
$$y(\theta) = \sin\theta$$

Looking at start, end we see

$$\pi/2 \leq \theta < \pi$$

What about a t parametrization? Well, exp-ing $t = \ln \tan \theta/2$, we get

$$e^t = \tan \theta/2$$

We now have to solve for $\sin\theta$ and $\cos\theta$ in terms of $\tan \theta/2$.

This is a trig exercise very similar to what we've already done.

Recall

$$\sin\theta = 2\sin\theta/2\,\cos\theta/2$$

$$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$$

and that we can use the sin and cos in terms of tan formulas to write

$$\sin\theta/2 = \frac{\tan\theta/2}{\sqrt{1+\tan^2\theta/2}}$$

$$\cos\theta/2 = \frac{1}{\sqrt{1+\tan^2\theta/2}}$$

so we have

$$\sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}$$

Now plugging in $e^t = \tan\theta/2$,

we get

$$\sin\theta = \frac{2e^t}{1+e^{2t}} = \frac{2}{e^{-t}+e^t} = \operatorname{sech} t$$

and

$$\cos\theta = \frac{1-e^{2t}}{1+e^{2t}} = \frac{e^{-t}-e^t}{e^{-t}+e^t} = -\tanh t$$

so we can also parametrize the tractrix by

$$(t - \tanh t, \operatorname{sech} t), \; t \geq 0.$$

Parametrized curves, examples and constructions

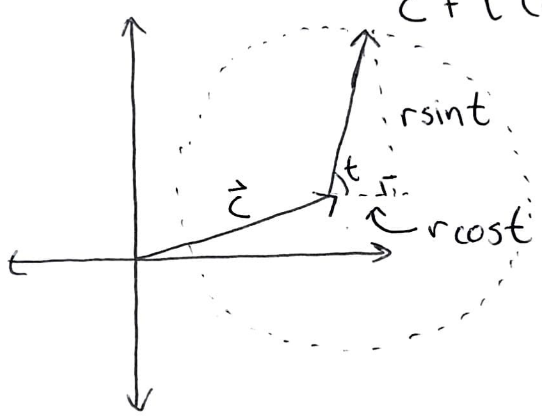
Recall that a parametrized curve is a map $\vec{\alpha}: \mathbb{R} \to \mathbb{R}^n$. We will now study some example curves.

Example. The circle of radius r with center $\vec{c} = (c_1, c_2)$ in $\mathbb{R}^2$ is described implicitly bey

$$(x - c_1)^2 + (y - c_2)^2 = r^2$$

We can parametrize this curve by

$$\vec{c} + r(\cos t, \sin t) = \vec{\alpha}(t)$$



Notice that

$$\vec{\alpha}(t) = (c_1 + r\cos t,\ c_2 + r\sin t)$$

obeys

$$(\alpha_1(t) - c_1)^2 + (\alpha_2(t) - c_2)^2 =$$

$$= (r\cos t)^2 + (r\sin t)^2$$

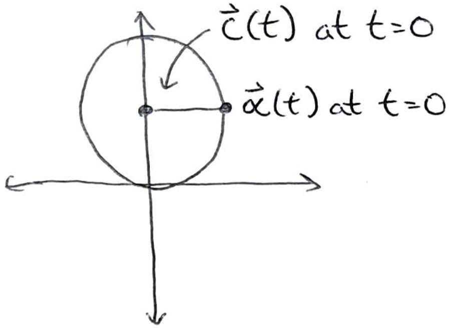
$$= r^2(\cos^2 t + \sin^2 t) = r^2,$$

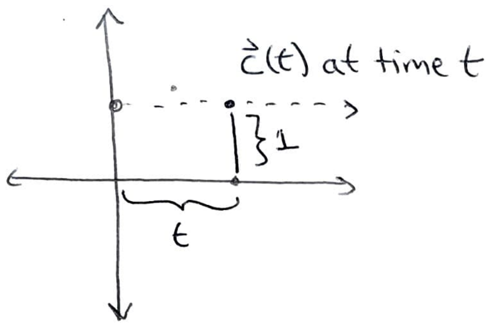
but there is more information in the ~~para~~ parametrization $\vec{\alpha}(t)$ because it tells us <u>when</u> each point on the circle is reached.

Example 2. $\vec{\alpha}(t) = (c_1 + r\cos(t^2), c_2 + r\sin(t^2))$ also parametrizes the circle of radius r and center $\vec{c} = (c_1, c_2)$.

We can make some beautiful curves by combining sines and cosines.

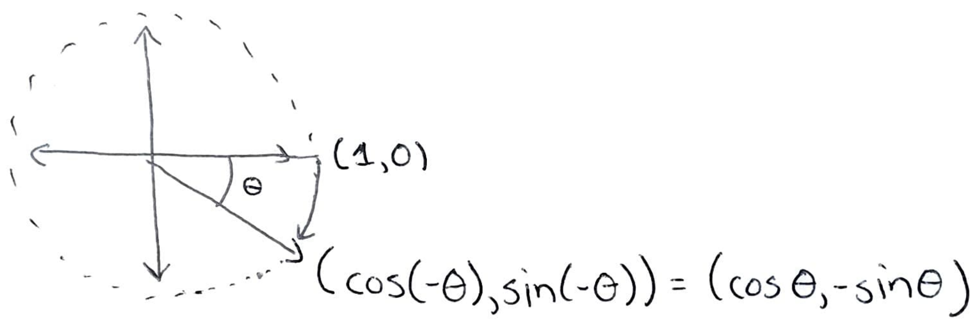
Example. A unit circle starts with center at $(0,1)$ and rolls along the pos. x axis. Parametrize the path of a point starting at $(1,1)$.





If the center of the circle is given by $\vec{c}(t)$, we can assume that the circle is rolling to the right at unit speed, so $\vec{c}(t) = (t,1)$.

However, if ~~th~~ a unit circle has rolled t units forward, it has turned by ~~an~~ angle of t radians... in the <u>clockwise</u> direction.



This rotation carries the point at $(1,0)$ ~~to the~~ (relative to the center) to the point at $(\cos\theta, -\sin\theta)$ (relative to the center).

Adding these together:

$$\vec{\alpha}(t) = (t + \cos t, 1 - \sin t)$$

We will work through a more elaborate ~~ex~~ example of this type of motion in homework when we describe the square-wheeled car.

We often describe curves with a differential equation, so let's remember how to solve (easy) ODEs.

If $u'(t) = F(u(t))$, then

$$\frac{u'(t)}{F(u(t))} = 1$$

integrate w.r.t. t

$$\int \frac{u'(t)}{F(u(t))} dt = \int 1 \, dt$$

$du = u'(t)\, dt$

$$\int \frac{1}{F(u)} du = \int 1 \, dt$$

so $\int {}^{1}/_{F(u)} \, du = t + C$.

If we can do the integral on the left to get some

$$G(u) = \int \frac{1}{F(u)} du$$

then we get an equation

$$G(u) = t$$

which we can try to solve for $u(t)$.

Example. $u'(t) = u(t)^2$.

$$\frac{u'(t)}{u(t)^2} = 1 \quad \Rightarrow \quad \int \frac{1}{u(t)^2} u'(t)\,dt = \int 1\,dt$$

$$\Rightarrow \int \frac{1}{u^2} du = t + C \quad \Rightarrow \quad -\frac{1}{u} = t + C$$

$$\Rightarrow \quad u = -\frac{1}{t+C}$$

So $u(t) = -\frac{1}{t+C}$, and indeed $u'(t) = \frac{1}{(t+c)^2} = u(t)^2$.