FACULTY OF SCIENCE
UNIVERSITY OF COPENHAGEN

Mads Chr. Hansen

# Morse Theory on Complex Grassmannians

Construction from the classical outset to Morse-Bott theory

Advisor: Prof. Nathalie Wahl

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**Abstract**

The purpose of the present paper is an investigation of classical Morse Theory, to examine topological aspects of complex Grassmannians. We will develop this theory from scratch, introducing the Hessian and Morse functions, proving the fundamental results of the Morse Lemma and the Morse principles, before giving the main result of Morse theory, ultimately telling us how to obtain a CW-complex structure on any smooth manifold. Also the Morse inequalities will be derived, giving us a connection between the homology of the underlying manifold, in form of the Betti numbers, and the critical points. We then give a couple of examples, before moving on to generalize the classical theory to Morse-Bott theory, allowing us to calculate the Poincaré polynomial of complex Grassmannians, by construction of a perfect Morse-Bott function. This turns out to be an efficient way to quantitatively describe smooth manifolds in general. Indeed, this polynomial contains enough topological information to completely determine the homology – and with some additional efforts even the cohomology – of the complex Grassmannians.

**Resumé**

Formålet med nærværende skrivelse er en undersøgelse af klassisk Morse teori, med henblik på at opnå indsigt i topologiske aspekter ved komplekse Grassmannians. Vi udvikler denne teori fra bunden, idet vi introducerer Hessianen og Morse funktioner, inden vi viser de fundamentale resultater i Morse lemmaet og Morse principperne, hvilket leder til hovedresultatet i Morse teori, som i sidste ende giver os en CW-kompleks dekomposition af enhver glat mangfoldighed. Derudover udledes Morse ulighederne, hvilket giver os en sammenhæng mellem homologien af den bagvedliggende mangfoldighed, i form af Betti-tallene, og de kritiske punkter. Herefter gives et par eksempler, før vi generaliserer den klassiske Morse teori til Morse-Bott teori, hvilket tillader os at beregne Poincaré-polynomiet af komplekse Grassmannians, ved konstruktion af en perfekt Morse-Bott funktion. Dette viser sig at være en effektiv måde hvorpå glatte mangfoldigheder kan beskrives kvantitativt, idet disse polynomier indeholder nok topologisk information til fuldstændigt at bestemme homologien – og med nogle yderligere bestræbelser sågar kohomologien – af komplekse Grassmannians.

# Preface

The present paper is the result of a small Masters Project, written at the Department of Mathematical Sciences, University of Copenhagen, through late spring 2012. I would like to thank my advisor, Professor Nathalie Wahl, first and foremost for her kind help and willingness to engage in the project on a rather short notice, and secondly for giving me an illuminating introduction to differential topology in general, through the course DiffTop taken just before this project was initiated. Furthermore, I thank Dr. Richard Hepworth for vividly introducing me to topology in the first place, and for his continuous support ever since.

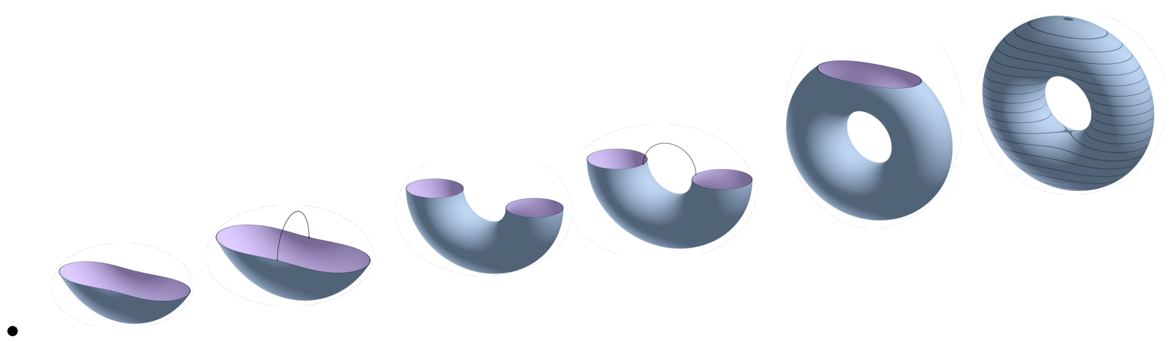
It is an elementary fact from topology that if $M$ is a compact manifold, then any continuous function $f$ on $M$ has maximum and minimum points, and that these are critical points. “Morse Theory” is the general term for a far-reaching and seemingly inexhaustible idea, greatly extending this result and thus connecting analysis, topology and geometry. More precisely, it is possible to determine topological properties of a finite or infinite dimensional manifold from the critical points of just one suitable function on the manifold. Indeed, one can for example construct CW-complex structures of *any* smooth manifold this way, and in some cases calculate the homology - or even cohomology - of this underlying manifold.

The subject started out in the 30’s with Marston Morse (1892-1977) and his calculus of variations in the large. Since then, the subject has seen many surprising extensions, to reemerge as an important source for grasping ever more diverse areas of mathematics. In the 40’s, Réné Thom (1923-2002), who would later be one of the founders of catastrophe theory, made contact with physics. In the 50’s, Raoul Bott (1923-2005) greatly generalized Morse theory to study homotopy theory of Lie groups, eventually leading to the celebrated Bott periodicity theorem. We shall also take a look at this extension, to be able to analyze Grassmannians. Stephen Smale (1930-), a Ph.D. student under Raoul Bott and later Fields medalist, made the next great leap for Mose theory in the 60’s by connecting it with dynamical systems. In the 70’s Edward Witten (1951-) rediscovered Morse theory through a supersymmetric approach, awarding him the Fields medal as well. Finally, in the 1980, Andreas Floer (1951-1991) invented an infinite dimensional analog of finite dimensional Morse homology, applicable to symplectic geometry and low-dimensional topology, which has also set the stage for present day research in topological field theory. This resilience of Morse theory is reflected in the very title of Bott’s survey article “Morse Theory Indomitable” [Bot88] in which a more detailed exposition of the development of Morse theory can be found. See also Guest’s paper “Morse Theory in the 1990’s” for a more modern and very readable account.

Morse theory is thus a huge subject connecting many interesting fields of mathematics, and, consequentially, has an enormous amount of applications. Among these are the Gauss-Bonnet theorem, the Poincaré-Hopf index theorem, Poincaré duality, determination of geodesics on a manifold, Lefschetz singularity of hypersurfaces, Yang-Mills theory on vector bundles, Milnor’s exotic spheres and Hamiltonian dynamics, to name but a few, see [Pet01]. Recently, there have also been numerous real world applications within robotics, crystallographics and image analysis.

Needless to say, this paper cannot cover even a tiny fraction of this. Instead, we aim to give a motivating introduction to the general idea of Morse theory for finite dimensional manifolds, by looking at the exceptional, and now classical, exposition of the subject by John Milnor [Mil68] with a few of the later developments by Bott and Smale, allowing us to look at an interesting example, the complex Grassmanians, and thereby displaying some of the strengths of Morse theory. The prerequisites for this paper is thus kept to a minimum, although the reader is assumed to be familiar with some differential topology, differential geometry and functional analysis.

Mads Christian Hansen<br>Copenhagen, Denmark<br>June 20, 2012



# Contents

# 1 Classical Morse Theory

In this section we will introduce the concepts of what has become known as the somewhat classical fields of Morse Theory. We will mainly follow the outline by John Milnor [Mil68], which has become a classic on its own [Gue02]. More specifically, we will introduce the notions of critical points, index, Morse functions, prove the Lemma of Morse and discuss the existence of Morse functions. After covering some background material on dynamics, we will move on to prove the Morse principles, leading to the main theorem which will give us a surprising connection between the shape of a manifold and the functions defined on it.

## 1.1 Morse functions

Our analysis will be carried out on certain particularly nice functions, called Morse functions. We will later see that this restriction is not too harsh, indeed the space of Morse functions is dense in the space of functions. Yet being a Morse function is paradoxically also a strong condition since these will have a very special local canonical form. One could perhaps take this as an indication of a good definition [Gue02].

We first need to introduce some terminology. In all what follows, we will let $M$ be an $m$-dimensional smooth manifold. The tangent space at a point $p \in M$ will be denoted by $T_pM$. If $f : M \to M'$ is a smooth map with $f(p) = q$, then the induced map, the differential of $f$ at $p$, will be denoted by $df_p : T_pM \to T_qM'$. We can now define the very central concept of a point being critical.

**Definition 1.1.1.** Let $f : M \to M'$ be a smooth map between smooth manifolds. A point $p \in M$ is *critical* if the differential $df_p$ is not surjective. A point $q \in M'$ is a *critical value* if the fibre $f^{-1}(q)$ contains a critical point of $f$. We denote by $Cr_f$ the set of critical points for $f$. $\triangleleft$

**Note 1.** In particular, for a smooth map $f : M \to \mathbb{R}$, the point $p \in M$ is critical if $df_p : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is zero. Introducing local coordinates $(x_1, \dots, x_m)$ in a neighborhood $U$ of $p$, this is equivalent to

$$\frac{\partial f}{\partial x_1}(p) = \cdots = \frac{\partial f}{\partial x_m}(p) = 0, \tag{1.1}$$

where we use the convention of treating coordinate maps as identifications, thus simplifying the expression $\frac{\partial (f\circ\varphi^{-1})}{\partial x_i}(\varphi(p))$, where $\varphi : U \to \mathbb{R}^m$ is a smooth chart, see [Lee02, Chapter 3] for more details. This convention will be followed throughout the rest of this project. $\diamondsuit$

**Example 1.1.1.** Following [Nic07, Example 1.4], let $M$ be an $m$-manifold embedded in a Euclidean space $E$ and $f : E \to \mathbb{R}$ a smooth function. Note that by [BJ82, Whitneys embedding theorem] we know that such an embedding always exists. A point $p \in M$ is critical for the restricted map $f|_M$ if the following holds, where $\langle\cdot,\cdot\rangle$ denotes the canonical inner product in $E$:

$$\langle \nabla f(p), v\rangle = 0, \qquad \forall v \in T_pM. \tag{1.2}$$

Referring back to a basic course on Calculus of several variables, we know that the gradient at $p$ is orthogonal to the tangent space of the level set containg $f(p)$. Therefore, by [Note 1] and (1.2), $p \in M$ is critical for $f|_M$ if $p$ is a critical point of $f$, or $\nabla f(p) \neq 0$ and $T_pM$ is contained in $T_pf^{-1}(f(p))$. Now, if $f$ is a non-zero linear function, then all its level sets are hyperplanes perpendicular to a common vector $v \in E$, and $p \in M$ is a critical point of $f|_M$ exactly if $v \perp T_pM$.

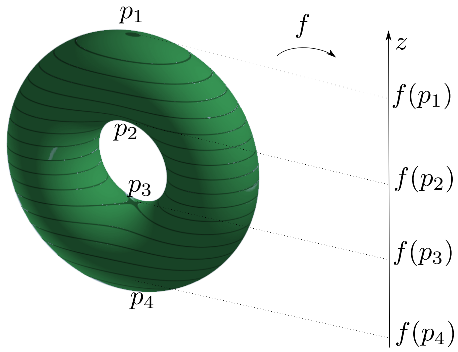



Figure 1.1: The height function on $T^2$ embedded in $\mathbb{R}^3$.

In particular, the height function $f : \mathbb{R}^3 \to \mathbb{R}$ given by $f(x, y, z) = z$ is such a non-zero linear function, with its level sets perpendicular to $v = (0, 0, 1)$, that is horizontal planes. Applying this to the classical example of the torus $T^2$ embedded vertically in $\mathbb{R}^3$, we see that there are exactly four critical points. Indeed, in the above figure, $p_1, \ldots, p_4$ are critical points, while $f(p_1), \ldots, f(p_4)$ are the corresponding critical values. $\diamondsuit$

### 1.1.1 The Hessian

We will need to distinguish between two kinds of critical points, what we will later call degenerate and non-degenerate critical points, since the first type is too unstable when considering topological questions. Indeed, if we perturb a function with a non-degenerate critical point slightly, we will obtain another non-degenerate critical point close to the first one, while perturbing a function with a degenerate critical point could result in either loosing or obtaining a critical point, see [Mat02]. This needs to be ruled out. To be able to make this distinguishing, however, we will need to introduce the Hessian, following [Nic07].

Recall that, by definition, a vector field on a manifold $M$ is an assignment of a tangent vector $Y_p \in T_pM$ to each $p \in M$. It is called smooth, if to any smooth coordinate chart $(U, (x^i))$ around $p$,

$$Y_p = \sum_{i=1}^{m} a_i(p) \frac{\partial}{\partial x^i}\Big|_p \tag{1.3}$$

for all $p \in U$, where the $m$ component functions $Y^i \in C^\infty(U)$ are smooth, see for example [Sch07].

**Definition 1.1.2** (Hessian)**.** Let $p_0$ be a critical point of the smooth function $f : M \to \mathbb{R}$. The *Hessian* of $f$ at $p_0$ is the map

$$H_{f,p_0} : T_{p_0}M \times T_{p_0}M \to \mathbb{R} \qquad \text{given by} \qquad H_{f,p_0}(v, w) = (XYf)(p_0) \tag{1.4}$$

where $X, Y$ are vector fields on $M$ such that $X(p_0) = v, Y(p_0) = w$. $\triangleleft$

This is well defined by the following fact, which we recall from differential geometry: Let $X$ and $Y$ be smooth vector fields on a smooth manifold $M$. Given a smooth function $f : M \to \mathbb{R}$, we can apply $Y$ to $f$ and obtain another smooth function $Yf$, cf. [Lee02, Lemma 4.6]. In turn we can apply $X$ and obtain yet another smooth function $XYf = X(Yf)$. Thus $H_{f,p_0}$ is well defined. While we are at it, let's denote the directional derivative of $f$ at $p$ in direction $v \in \mathbb{R}^m$

$$D_{p,v}(f) = \frac{d}{dt}\Big|_{t=0} f(p + tv) \in \mathbb{R} \tag{1.5}$$

see for example [Sch07]. With this in mind, we can now prove the following lemma, implying that the Hessian is invariant under coordinate changes.

**Lemma 1.1.1.** *The Hessian $H_{f,p_0}$ is a bilinear and symmetric functional independent of the choice of vector fields $X, Y$ extending $v, w$.*

*Proof.* Suppose $p_0$ is a critical point where $X(p_0) = X'(p_0)$ and $Y(p_0) = Y'(p_0)$, for some vector fields $X', Y'$ extending $v, w$ respectively as well. Since the Lie bracket of two smooth vector fields is again a smooth vector field, cf. [Sch07, Theorem 6.5] we obtain:

$$(XY - YX)f(p_0) = [X,Y]f(p_0) = D_{p_0,[X,Y](p_0)}f = 0 \tag{1.6}$$

where the last equation follows because $p_0$ is a critical point. Therefore $(XYf)(p_0) = (YXf)(p_0)$ and hence $H_{f,p_0}$ is symmetric. Since $(X - X')(p_0) = 0$ we see that

$$(X - X')g(p_0) = D_{p_0,(X-X')(p_0)}g = 0 \qquad \forall g \in C^\infty(M) \tag{1.7}$$

In particular, since $Yf \in C^\infty(M)$ cf. [Sch07, Lemma 6.2.2], we infer that $(X - X')Yf(p_0) = (XYf)(p_0) - (X'Yf)(p_0)$, which in turn gives us

$$(XYf)(p_0) = (X'Yf)(p_0) = (YX'f)(p_0) = (Y'X'f)(p_0) = (X'Y'f)(p_0). \tag{1.8}$$

We conclude that $H_{f,p_0}$ is independent of the chosen vector field as required. Finally, to see that it is bilinear, let $\hat{Y}, \tilde{Y}$ be vector fields such that $\hat{Y}(p_0) = Y_0$ and $\tilde{Y}(p_0) = Y_1$. Then for $a_0, a_1 \in \mathbb{R}$, $a_0\hat{Y} + a_1\tilde{Y}$ is also a smooth vector field on $M$. Now we get

$$\begin{aligned} H_{f,p_0}(X_0, a_0Y_0 + a_1Y_1) &= H_{f,p_0}(X_0, a_0Y_0 + a_1Y_1) = X(a_0\hat{Y} + a_1\tilde{Y})f(p_0) && (1.9)\\ &= a_0X\hat{Y}f(p_0) + a_1X\tilde{Y}f(p_0) = a_0H_{f,p_0}(X_0, Y_0) + a_1H_{f,p_0}(X_0, Y_1) && (1.10) \end{aligned}$$

and similarly for the first coordinate. We conclude that $H_{f,p_0}$ is a bilinear functional as desired. □

**Definition 1.1.3.** A critical point $p_0$ of a smooth function $f : M \to \mathbb{R}$ is called *non-degenerate* if its Hessian is non-degenerate, that is

$$H_{f,p_0}(v, w) = 0 \quad \forall w \in T_{p_0}M \qquad \Leftrightarrow \qquad v = 0 \tag{1.11}$$

A smooth function is called a *Morse function* if all its critical points are non-degenerate. ◁

**Note 2.** In the basis $\frac{\partial}{\partial x^i}|_{p_0}, \dots, \frac{\partial}{\partial x^m}|_{p_0}$ for the tangent space $T_{p_0}M$, the Hessian is represented by the matrix

$$H_{ij}(p_0) = \Big[\frac{\partial^2 f}{\partial x^i \partial x^j}(p_0)\Big]_{1\le i\le m, 1\le j\le m} \tag{1.12}$$

Furthermore, $p_0$ is non-degenerate if and only if $\det H_{ij}(p_0) \neq 0$. Indeed, we can write the smooth vector fields locally as follows:

$$X = \sum_{i=1}^m a_i \frac{\partial}{\partial x^i} \qquad \text{and} \qquad Y = \sum_{j=1}^m b_j \frac{\partial}{\partial x^j} \tag{1.13}$$

where $a_i, b_j \in C^\infty(M)$ are the components of $X$ and $Y$ respectively, cf.[Lee02, p. 83-84]. We can assume that the $b_j$'s are constant by [Lemma 1.1.1]. Then we have

$$\begin{aligned} H_{f,p_0}(v, w) &= (XYf)(p_0) = X(Yf)(p_0) = \sum_{i=1}^m a_i \frac{\partial}{\partial x^i}\Big(\sum_{j=1}^m b_j \frac{\partial f}{\partial x^j}\Big)(p_0) && (1.14)\\ &= \sum_{i=1}^m a_i\Big(\sum_{j=1}^m \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x^j} + b_j \frac{\partial^2 f}{\partial x^i \partial x^j}\Big)(p_0) = \sum_{i,j=1}^m a_i(p_0)b_j(p_0)\frac{\partial^2 f}{\partial x^i \partial x^j}(p_0) && (1.15)\\ &= [a_i(p_0)]H_{ij}(p_0)[b_j(p_0)] && (1.16) \end{aligned}$$

Suppose $H_{f,p_0}(v, w) = 0$ for some $w \neq 0$. Then $w = Y(p_0) = \sum_{j=1}^m b_j(p_0)\frac{\partial}{\partial x_j}|_{p_0}$ thus there exists $b_i \in C^\infty(M)$ such that $b_j(p_0) \neq 0$. If $\det(H_{ij}(p_0)) \neq 0$, then $H_{ij}(p_0)[b_j(p_0)] \neq [0]$ hence there exists a function $a_i \in C^\infty(M)$ such that $a_i(p_0) = 0$. Now if $H_{f,p_0}(v, w) = 0$ for all $w \in T_{p_0}M$, then $a_i(p_0) = 0$ for all $i$, that is $v = X(p_0) = \sum_{i=1}^m a_i(p_0)\frac{\partial}{\partial x^i}|_{p_0} = 0$. Thus, if $\det(H_{ij}(p_0)) \neq 0$, then $p_0$ is non-degenerate. On the other hand, suppose $\det(H_{ij}(p_0)) = 0$, then there exists some $[b_j(p_0)]$ such that $H_{ij}(p_0)[b_j(p_0)] = [0]$ and thus $H_{f,p_0}(v, w) = 0$ also for $v \neq 0$. Therefore, by contraposition, if $p_0$ is non-degenerate, $\det(H_{ij}(p_0)) \neq 0$. ♢

#### Examples of degenerate and non-degenerate critical points

In [Figure 1.1.1] we see different interesting cases of functions on $\mathbb{R}^n$. In the first, the function $f(x,y) = x^2 + y^2$ has the origin as a non-degenerate critical point. In the second, the function $f(x,y) = x^3 - 3xy^2 = \mathrm{Re}(x+iy)^3$ has the origin as a degenerate critical point. Finally, the last case shows $f(x,y) = x^2$ which has the entire $x$-axis as degenerate critical points, cf. [Mil68].

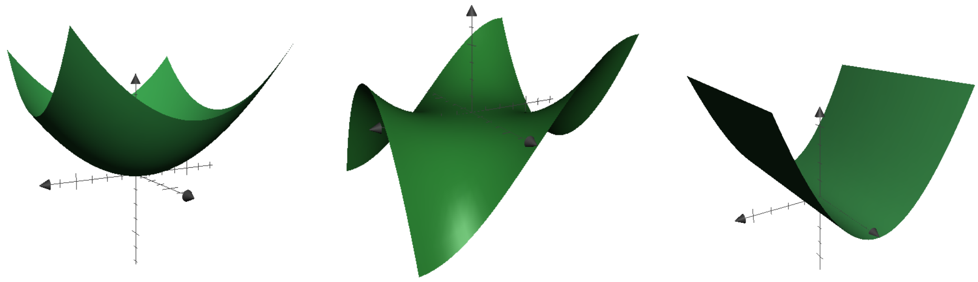


Figure 1.2: Examples of critical points for functions on $\mathbb{R}^2$.

### 1.1.2 The Lemma of Morse

Using the Hessian, we will in this section describe the local structure of Morse functions, following [Mil68] and [Mat02]. Recall, that with the *index* of a bilinear functional $H$ on a vector space $V$, we understand the maximal dimension of a subspace of $V$ on which $H$ is negative definite. We can now define the index of a smooth function at a non-degenerate critical point.

**Definition 1.1.4.** Let $p_0$ be a non-degenerate critical point of a smooth function $f : M \to \mathbb{R}$. Its *index*, $\lambda(p_0)$, is the index of the Hessian $H_{f,p_0}$. ◁

One could equivalently define the index of $p_0$ as the number of negative eigenvalues of the Hessian matrix, $H_{ij}(p_0)$, see [Bot88]. This index contains surprisingly much information. In fact, the lemma of Morse below shows that the behavior of $f$ in a neighborhood of a point $p$, can be completely described by this index. However, we need the following preliminary lemma:

**Lemma 1.1.2.** *Let $U$ be a convex neighborhood around the origin in $\mathbb{R}^m$, and $f : U \to \mathbb{R}$ a smooth function. Then there exists smooth functions $g_1, \dots, g_m : U \to \mathbb{R}$ such that*

$$f(x) = f(0) + \sum_{i=1}^{m} x_i g_i(x_1, \dots, x_m) \tag{1.17}$$

*Proof.* By the fundamental theorem of calculus we have

$$f(x) - f(0) = \int_0^1 \frac{d}{dt} f(tx_1, \dots, tx_m)\, dt = \sum_{i=1}^{m} x_i \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m)\, dt \tag{1.18}$$

Thus we can define $g_i(x_1, \dots, x_m) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \dots, tx_m)\, dt$. □

**Lemma 1.1.3** (Morse Lemma)**.** *Let $p_0$ be a non-degenerate critical point of $f : M \to \mathbb{R}$. Then there is a local coordinate system $(x_1, x_2, \dots, x_m)$ around $p_0$ with $x_i(p_0) = 0$ for all $i$ and such that the indentity:*

$$f = f(p_0) - x_1^2 - x_2^2 - \dots - x_\lambda^2 + x_{\lambda+1}^2 + \dots + x_m^2 \tag{1.19}$$

*holds in this neighborhood, where $\lambda$ is the index of $f$ at $p_0$.*

*Proof.* Suppose first, that such an expression (1.19) for $f$ exists. Then $\lambda$ must be the index. Indeed for any coordinate system $(z_1, \dots, z_m)$, if

$$f(q) = f(p_0) - z_1(q)^2 - z_2(q)^2 - \dots - z_\lambda(q)^2 + z_{\lambda+1}(q)^2 + \dots + z_m(q)^2 \tag{1.20}$$

then we have

$$\frac{\partial^2 f}{\partial z_i, \partial z_j}(p_0) = \begin{cases} 2 & \text{if } i = j > \lambda \\ -2 & \text{if } i = j \leq \lambda \\ 0 & \text{if } i \neq j \end{cases} \tag{1.21}$$

thus the matrix representing $H_{f,p_0}$ with respect to the basis $\frac{\partial}{\partial z_1}|_{p_0}, \dots, \frac{\partial}{\partial z_m}|_{p_0}$ is the diagonal matrix

$$\begin{bmatrix} -2 & & & & & \\ & \ddots & & & & \\ & & -2 & & & \\ & & & 2 & & \\ & & & & \ddots & \\ & & & & & 2 \end{bmatrix} \tag{1.22}$$

We see that there is a subspace, $V_-$ of $\mathbb{R}^m$ of dimension $\lambda$ where $H_{f,p_0}$ is negative definite, and a subspace $V_+$ of $\mathbb{R}^m$ of dimension $m - \lambda$ where $H_{f,p_0}$ is positive definite. If there were a subspace of $\mathbb{R}^m$ of dimension greater than $\lambda$ on which $H_{f,p_0}$ was negative definite, then this would intersect with $V_+$, which is impossible.

We now show that a suitable coordinate system $(x_1, \dots, x_m)$ exists. We can assume that $p_0$ corresponds to the origin $(0, \dots, 0)$, and furthermore that $f(p_0) = f(0) = 0$, replacing $f$ by $f - f(p_0)$ if necessary. By [Lemma 1.1.2] there exists smooth functions $g_1, \dots, g_m$ in a neighborhood of the origin such that

$$f(x_1, \dots, x_m) = \sum_{j=1}^{m} x_j g_j(x_1, \dots, x_m). \tag{1.23}$$

From the proof of [Lemma 1.1.2] it furthermore follows that

$$g_i(\mathbf{0}) = \int_0^1 \frac{\partial f}{\partial x_i}(\mathbf{0})\, dt = \frac{\partial f}{\partial x_i}(\mathbf{0}) = 0 \tag{1.24}$$

where the last equality follows since $p_0 = \mathbf{0}$ is assumed to be a critical point. Therefore we can apply [Lemma 1.1.2] once again to $g_j$ hence obtaining

$$g_j(x_1, \dots, x_m) = \sum_{i=1}^{m} x_i h_{ij}(x_1, \dots, x_m) \tag{1.25}$$

for some smooth functions $h_{ij}$ in a neighborhood of the origin. It follows that

$$f(x_1, \dots, x_m) = \sum_{i,j=1}^{m} x_i x_j h_{ij}(x_1, \dots, x_m). \tag{1.26}$$

Letting $\overline{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji})$ we get

$$f(x_1, \dots, x_m) = \frac{1}{2}\Big(\sum_{i,j=1}^{m} x_i x_j h_{ij}(x_1, \dots, x_m) + \sum_{i,j=1}^{m} x_i x_j h_{ji}(x_1, \dots, x_m)\Big) \tag{1.27}$$

$$= \sum_{i,j=1}^{m} x_i x_j \Big(\frac{1}{2} h_{ij}(x_1, \dots, x_m) + h_{ji}(x_1, \dots, x_m)\Big) = \sum_{i,j=1}^{m} x_i x_j \overline{h}_{ij}(x_1, \dots, x_m) \tag{1.28}$$

and furthermore $\overline{h}_{ij} = \overline{h}_{ji}$. Note that by the chain rule we have

$$\frac{\partial f}{\partial x_i \partial x_j}(\mathbf{x}) = \overline{h}_{ij}(\mathbf{x}) + x_j \frac{\partial \overline{h}_{ij}}{\partial x_j}(\mathbf{x}) + x_i \frac{\partial \overline{h}_{ij}}{\partial x_i}(\mathbf{x}) + x_i x_j \frac{\partial^2 \overline{h}_{ij}}{\partial x_i \partial x_j}(\mathbf{x})$$

$$+ \overline{h}_{ji}(\mathbf{x}) + x_j \frac{\partial \overline{h}_{ji}}{\partial x_j}(\mathbf{x}) + x_i \frac{\partial \overline{h}_{ji}}{\partial x_i}(\mathbf{x}) + x_i x_j \frac{\partial^2 \overline{h}_{ji}}{\partial x_i \partial x_j}(\mathbf{x}) \tag{1.29}$$

and therefore, since $p_0$ is assumed to be a non-degenerate critical point, we obtain

$$\det[\overline{h}_{ij}(\mathbf{0})]_{ij} = \det\Big[\frac{1}{2}(\overline{h}_{ij}(\mathbf{0}) + \overline{h}_{ji}(\mathbf{0}))\Big]_{ij} = \det\Big[\frac{1}{2}\frac{\partial f}{\partial x_i \partial x_j}(\mathbf{0})\Big]_{ij} \neq 0 \tag{1.30}$$

The idea of the proof is now to change the representation of $f$ in (1.27), which is a generalized version of a quadratic form, to the form in (1.19), which is a genuine special quadratic form, by induction on the number of terms in the generalized quadratic form of $f$. Assume therefore, that there exists coordinates $u_1, \dots u_m$ in a neighborhood $U_1$ of 0 such that

$$f = \pm u_1^2 + \cdots \pm u_{k-1}^2 + \sum_{i,j=k}^{m} u_i u_j H_{ij}(u_1, \dots, u_m) \tag{1.31}$$

in $U_1$, where the matrices $[H_{ij}(u_1, \dots, u_m)]_{ij}$ are symmetric. After a linear change in the last $m - k + 1$ coordinates we may assume that $\frac{\partial^2 f}{\partial x_k^2}(\mathbf{0}) \neq 0$, and hence by (1.30) that $H_{kk}(\mathbf{0}) \neq 0$. Since $H_{kk}$ is continuous, there exists a neighborhood $U_2 \subset U_1$ of 0 where $H_{kk}$ is bounded away from zero. Therefore we can introduce new smooth coordinates in yet another neighborhood $U_3 \subset U_2$:

$$v_k = \sqrt{|H_{kk}|}\Big(u_k + \sum_{i=k+1}^{m} u_i \frac{H_{ik}}{H_{kk}}\Big) \qquad \text{and} \qquad v_i = u_i \quad \text{for } i \neq k. \tag{1.32}$$

Now we note that if we take the square of this coordinate function we obtain

$$v_k^2 = \pm u_k^2 H_{kk} \pm 2 \sum_{i=k+1}^{m} u_k u_i H_{ik} \pm (\sum_{i=k+1}^{m} u_i H_{ik})^2 / H_{kk} \tag{1.33}$$

with $+$ all places if $H_{kk} > 0$ and $-$ all places if $H_{kk} < 0$ in this neighborhood. Comparing this with the expression (1.31) we see that

$$f = \pm u_1^2 + \cdots + u_k^2 + \sum_{i,j=k+1}^{m} v_i v_j H_{ij} - \Big(\sum_{i=k+1}^{m} v_i H_{ki}\Big)^2 / H_{kk} = \pm v_1^2 \pm \cdots \pm v_k^2 + \sum_{i,j=k+1}^{m} v_i v_j H'_{ij}$$

where the last equality follows, since the two last terms are sums over $v_{k+1}, \dots, v_m$ only. Since $M$ is finite dimensional, we conclude by induction that $f$ can be given the required expression. ☐

**Corollary 1.1.1.** *A non-degenerate critical point is isolated.*

*Proof.* Suppose $p_0$ is a non-degenerate critical point of $f : M \to \mathbb{R}$. By [Morse lemma], there is a local coordinate system $(x_1, \dots, x_m)$ around $p_0$, such that $f$ has the form (1.19) in this neighborhood. But functions of this type has no other critical points. ☐

**Corollary 1.1.2.** *A Morse function on a compact manifold admits only finitely many critical points.*

*Proof.* Suppose for contradiction that the given Morse function has infinitely many critical points, $p_1, p_2 \dots$,. We claim, that this sequence has a convergent subsequence $p_{n_1}, p_{n_2}, \dots$. Indeed, since by [BJ82, Whitney's Theorem] we can embed $M$ in $\mathbb{R}^N$, thus $M$ is metrizable. The claim now follows because every compact metrizable space is sequentially compact, cf. [Mun00, Theorem 28.2]. Let $p_0$ be the limit point[1] of the sequence $(p_{n_i})_{i \geq 1}$. Consider a local coordinate system $(x_1, \dots, x_m)$ in a neighborhood $U$ around $p_0$. By choosing a new subsequence if necessary, we may assume that $(p_{n_i})_{i \geq 1} \subset U$. Now, since $f \in C^\infty(M)$ the functions $\frac{\partial f}{\partial x_j} : M \to \mathbb{R}$ are continuous and since the sequence $(p_{n_i})_{i \geq 1}$ consists of critical points we get

$$\frac{\partial f}{\partial x_j}(p_0) = \lim_{i \to \infty} \frac{\partial f}{\partial x_j}(p_{n_i}) = \lim_{i \to \infty} 0 = 0 \qquad \text{for all } j \tag{1.34}$$

thus $p_0$ is a critical point, cf. [Note 1]. However, since $f$ is a Morse function, all critical points are non-degenerate and hence isolated by [Corollary 1.1.1], which is a contradiction. ☐

[1] Note that we are allowed to speak about *the* limit point since these are unique in a Hausdorff space. In particular, these are unique in $M$

**Definition 1.1.5.** If $f : M \to \mathbb{R}$ is a Morse function with finitely many critical points, $p_1, \dots, p_k$, we define the *Morse polynomial* of $f$ to be

$$P_f(t) = \sum_{p \in Cr_f} t^{\lambda(p)} = \sum_{i=1}^{k} t^{\lambda_i} = \sum_{\lambda \geq 0} C_\lambda t^\lambda \tag{1.35}$$

where $\lambda_i$ is the index of the critical point $p_i$ and $C_\lambda$ is the number of critical points of index $\lambda$. The coefficients are called the *Morse numbers* of the Morse function. $\triangleleft$

In view of [Corollary 1.1.2], the Morse polynomial is well defined on any compact manifold. Note furthermore, that $P_f(1)$ is the number of critical points for $f$. As an example, we can look at the following figure:

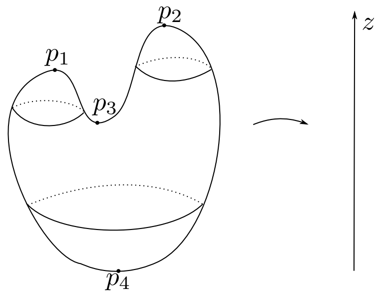



Figure 1.3: The height function restricts to a Morse function on the hypersurface, with four critical points, $p_1, \dots, p_4$, cf. [Example 1.1.1]. The Morse polynomial is $P_f(t) = 2t^2 + t + 1$.

### 1.1.3 Existence of Morse functions

Of course, all of the theory developed so far would be utterly useless, if Morse functions did in fact not exist. Luckily, this is not the case. On the contrary the set of Morse functions actually forms a dense subset of the space of smooth functions. Since the derivation of this is rather lengthy, we will only state the result, with the aim to understand exactly in which way we should interpret this existence. For the full proof, the reader is advised to consult [Mil68], [Mat02] and [Nic07] who all have detailed expositions of this matter.

If we embed a smooth manifold $M$ in $\mathbb{R}^N$, which is possible by [BJ82, Whitney's embedding theorem], one can show that the function $L_p : M \to \mathbb{R}$ given by $L_p(q) = \|p - q\|^2$ is a Morse function for almost all $p \in \mathbb{R}^N$, cf. [Mil68]. Here the term "almost all" means all but a set of measure zero, and comes from the application of Sard's theroem, see [BJ82] and [Mil65] for a definition of measure zero as well as proofs. The existence theorem claims that for any smooth function $g : M \to \mathbb{R}$ there exits a Morse function $f : M \to \mathbb{R}$ "arbitrarily close" to $g$. Since there are plenty of smooth functions on any smooth manifold around – take the constant ones for example – this would indeed guarantee the existence of Morse functions. In order to explain what is meant by "arbitrarily close", however, we need the following definition:

**Definition 1.1.6.** The function $f$ is a $(C^2, \epsilon)$-approximation of $g$ on the compact set $K$, contained in a coordinate neighborhood, if at every $p \in K$ we have

$$|f(p) - g(p)| < \epsilon, \qquad \left|\frac{\partial f}{\partial x_i}(p) - \frac{\partial g}{\partial x_i}(p)\right| < \epsilon, \qquad \left|\frac{\partial^2 f}{\partial x_i \partial x_j}(p) - \frac{\partial^2 g}{\partial x_i \partial x_j}(p)\right| < \epsilon \tag{1.36}$$

for $i, j = 1, \dots, m$. If $M$ is compact, we have a compact cover $M = \bigcup_{i=1}^{k} K_i$, each belonging to some coordinate neighborhood, and a function $f : M \to \mathbb{R}$ is a $(C^2, \epsilon)$-approximation to a function $g : M \to \mathbb{R}$, if $f$ is a $(C^2, \epsilon)$-approximation of $g$ on every $K_i$. $\triangleleft$

We can now summarize the above discussion into the existence theorem of Morse functions, whose proof, in various disguises, we encourage the reader to study in [Mil68], [Mat02] or [Nic07].

**Theorem 1.1.1** (Existence)**.** *Let $M$ be a smooth $m$-manifold. There exists an exhaustive Morse function on $M$. If furthermore $M$ is closed and $g : M \to \mathbb{R}$ is a smooth function defined on $M$, then there exists a Morse function $f : M \to \mathbb{R}$ which is $(C^2, \epsilon)$-close to $g : M \to \mathbb{R}$.*

## 1.2 Vector fields and dynamics

Before we move on to the main theorems of classical Morse theory, we will in this section cover some background material, which will be of great importance to the sequel. More specifically, we will introduce the notion of a flow on a manifold with the purpose of creating a so called gradient-like vector field on $M$. We will also cover some basic notions on Morse-Smale dynamics. Our investigation is inspired by [Mil68] and [Nic07].

### 1.2.1 Gradient-like vector fields

For ease of reference, let's recall again that a vector field on a manifold $M$ is an assignment of a tangent vector $Y_p \in T_pM$ to each $p \in M$. It is called smooth, if to any smooth coordinate chart $(U, (x^i))$ around $p$,

$$Y_p = \sum_{i=1}^{m} a_i(p) \frac{\partial}{\partial x^i}\Big|_p \tag{1.37}$$

for all $p \in U$, where the $m$ component functions $a_i \in C^\infty(U)$ are smooth, see for example [Sch07] or [Lee02]. We need a special type of such smooth vector fields taking the function $f$ defined on $M$ into account.

**Definition 1.2.1.** Let $M$ be a smooth manifold. A smooth map $\Phi : \mathbb{R} \times M \to M$ is called a *flow* on $M$, if for all $p \in M$ and all $t, s \in \mathbb{R}$ we have

$$\Phi(0, p) = p, \qquad \text{and} \qquad \Phi(t, \Phi(s, x)) = \Phi(t + s, x) \tag{1.38}$$

The map $\Phi_t : M \to M$ defined by $\Phi_t(p) = \Phi(t, p)$ defines a group homomorphism of $(\mathbb{R}, +)$ into $\mathrm{Diff}(M)$, the group of diffeomorphisms of $M$ onto itself, since we have $\Phi_s \circ \Phi_t = \Phi_{s+t}$ and $\Phi_0 = id_M$ and thus $\Phi_t^{-1} = \Phi_{-t}$. $\lhd$

Given a flow on $M$ and a smooth real valued function $f$, we define a vector field $X$ on $M$ by

$$X_p f = \lim_{h \to 0} \frac{f(\Phi_h(p)) - f(p)}{h} \tag{1.39}$$

and we say that $X$ generates the group $\Phi$, or that $X$ is the *velocity field* of the flow $\Phi$. The following lemma says that every vector field is the velocity field of exactly one flow, see [BJ82, Theorem 8.10] for an alternative proof.

**Lemma 1.2.1.** *A smooth vector field on $M$ which vanishes outside a compact set $K \subset M$ generates a unique flow on $M$.*

*Proof.* Given any smooth curve $\gamma : [0, 1] \to M$ we have the velocity vector $\frac{d\gamma}{dt} \in T_{\gamma(t)}M$, which as a derivation is given by $\frac{d\gamma}{dt}(f) = \lim_{h\to 0} \frac{f(\gamma(t+h)) - f(\gamma(t))}{h}$, cf. [BJ82, p. 76]. Let $\Phi$ be a flow generated by the smooth vector field $X$. For fixed $p$, the map $\alpha_p : \mathbb{R} \to M$ defined by $\alpha_p(t) = \Phi_t(p)$, is called the *integral curve* of $p$, see [BJ82, Definition 8.3]. We have

$$\begin{aligned}\frac{d\alpha_p(t)}{dt}(f) &= \frac{d\Phi_t(p)}{dt}(f) = \lim_{h\to 0} \frac{f(\Phi_{t+h}(p)) - f(\Phi_t(p))}{h} = \lim_{h\to 0} \frac{f(\Phi_h(\Phi_t(p))) - f(\Phi_t(p)}{h} \\ &= \lim_{h\to 0} \frac{f(\Phi_h(q)) - f(q)}{h} = X_q(f) = X_{\Phi_t(p)}(f) = X_{\alpha_p(t)}(f)\end{aligned} \tag{1.40}$$

where we have put $q = \Phi_t(p)$. The differential equation $\frac{d\alpha_p(t)}{dt} = X_{\alpha_p(t)}$ with initial condition $\alpha_p(0) = p$ has a unique solution, which depends smoothly on the initial condition, see for example [BJ82, p. 80-82]. Therefore, for each point in $M$ there exists a neighborhood $U$ and a number $\epsilon > 0$ such that this differential equation has a unique smooth solution for $p \in U$ and $|t| < \epsilon$.

Since $K$ is compact, we can cover it by finitely many such neighborhoods $U$. Let $\epsilon_0 > 0$ be the smallest of these corresponding $\epsilon$'s. Setting $\varphi(p) = p$ for $p \notin K$ we see that this differential equation has a unique smooth solution for $|t| \le \epsilon_0$ and all $p \in M$. Furthermore, assuming $|t|, |s|, |t + s| \le \epsilon_0$, we have $\varphi_s \circ \varphi_t = \varphi_{t+s}$ thus each $\varphi_t$ is a diffeomorphism.

To define $\varphi_t$ for $|t| \geq \epsilon_0$, note that for any $t \in \mathbb{R}$ we can write $t = k(\epsilon_0/2) + r$ for some $k \in \mathbb{Z}$ and a remainder $|r| < \epsilon/2$. We can then decompose $\varphi_t$ as follows, if $k \geq 0$:

$$\varphi_t = \varphi_{\epsilon_0/2} \circ \cdots \circ \varphi_{\epsilon_0/2} \circ \varphi_r \tag{1.41}$$

where $\varphi_{\epsilon_0/2}$ is composed with itself $k$ times. If $k < 0$ we compose $\varphi_{-\epsilon_0/2}$ with itself $k$ times. It now follows by construction that $\varphi_t$ is well defined and satisfies $\varphi_s \circ \varphi_t = \varphi_{s+t}$ as required. $\square$

**Definition 1.2.2.** Let $f : M \to \mathbb{R}$ be a Morse function. A smooth vector field $X$ on $M$ is said to be *gradient-like* for $f$, if it satisfies the following two conditions:

1. $X \cdot f > 0$ away from the critical points of $f$.
2. For every critical point $p_0$ of $f$ of index $\lambda$, there exists coordinates $(x^i)$ such that $x^i(p_0) = 0$, $f$ has the standard form (1.19) from the Morse lemma and

$$X = -2\sum_{i=1}^{\lambda} x^i \frac{\partial}{\partial x_i} + 2\sum_{i=\lambda+1}^{m} x^i \frac{\partial}{\partial x^i} \tag{1.42}$$

Note that this is indeed smooth by the definition given in the beginning. $\triangleleft$

Therefore, outside the critical points of $f$, a gradient-like vector field $X$ points in the direction into which $f$ is increasing. In particular, if $f$ is the height function, then $X$ points upward.

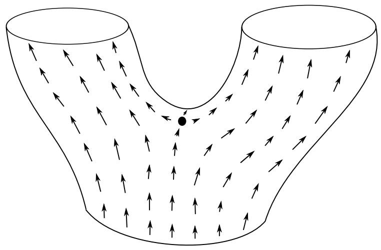


Figure 1.4: A gradient-like vector field.

**Lemma 1.2.2.** *Suppose $f : M \to \mathbb{R}$ is a Morse function on a smooth manifold $M$. Then there exists a gradient-like vector field $X$ for $f$.*

*Proof.* Choose a Riemannian metric, $g$, on $M$, which is always possible, cf. [Lee02, Proposition 11.26]. Then in any smooth local coordinates, $(x^i)$, we can write it as

$$g = g_{ij} dx^i \otimes dx^j = g_{ij} dx^i dx^j \tag{1.43}$$

where $(g_{ij})_{1 \leq i,j \leq m}$ is a symmetric positive definite matrix of smooth functions. In particular, since we are locally in $\mathbb{R}^m$, we can assume that this is the canonical Riemannian metric, such that $g = \delta_{i,j} dx^i dx^j = \sum_{i=1}^{m} (dx^i)^2$, cf. [Lee02, p. 274], and that in these coordinates $f$ has the form

$$f = f(p) - \sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2 \tag{1.44}$$

from the Morse lemma. Denote by $\nabla f \in$Vect$(M)$ the *gradient* of $f$ with respect to the metric $g$. This is defined to be the unique vector field that satisfies:

$$g(\nabla f, Z) = Zf \tag{1.45}$$

for every vector field $Z$. Now, in local coordinates we then have

$$\nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j} \tag{1.46}$$

where $(g^{ij})_{1\leq i,j\leq m}$ is the inverse matrix to $(g_{ij})_{1\leq i,j\leq m}$, see [Lee02, p. 283]. In particular with the canonical metric we have $\nabla f = \sum_{i=1}^m \frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^i}$. Note that this gives us $\nabla f \cdot f = \sum_{i=1}^m (\frac{\partial f}{\partial x^i})^2 \geq 0$ and that $(\nabla f \cdot f)(p) > 0$ if $p$ is not a critical point. Furthermore, for $f$ of the form in (1.44) we get, near a critical point $p_0$:

$$\nabla f = \sum_{j=1}^m \frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^j} = \sum_{j=1}^\lambda \frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^j} + \sum_{j=\lambda+1}^m \frac{\partial f}{\partial x^j}\frac{\partial}{\partial x^j} = -2\sum_{j=1}^\lambda x_j \frac{\partial}{\partial x^j} + 2\sum_{j=\lambda+1}^m x_j\frac{\partial}{\partial x^j} \tag{1.47}$$

thus $X = \nabla f$ satisfies the two requirements from [Definition (1.2.2)] and is therefore a gradient-like vector field. □

### 1.2.2 Stable and unstable manifolds

We will now cover some theory about Morse-Smale dynamics, following [Nic07]. Suppose $f : M \to \mathbb{R}$ is a Morse function on a compact manifold $M$ and $X$ a gradient-like vector field for $f$. Denote by $\Phi_t$ the flow on $M$ determined by $-X$.

**Lemma 1.2.3.** *For every $p_0 \in M$ the limits $\Phi_{\pm\infty}(p_0) = \lim_{t\to\pm\infty}\Phi_t(p_0)$ exists and are critical points of $f$.*

*Proof.* Given $p_0 \in M$, we have the integral curve $\alpha_{p_0}(t) = \Phi_t(p_0)$. If $\alpha_{p_0}(t)$ is the constant path, then we have $-X_{p_0}f = \lim_{t\to 0}\frac{f(\alpha_{p_0}(t))-f(p_0)}{t} = 0$ by (1.39) hence $p_0$ is a critical point as desired, cf. [Definition 1.2.2]. Now, suppose $\alpha_{p_0}(t)$ is not constant. From (1.40) and the fact that $X \cdot f \geq 0$ we deduce

$$\dot{f}(t) := \frac{d}{dt}f(\alpha_{p_0}(t)) = D_{\alpha_{p_0}(t),\frac{d}{dt}\alpha_{p_0}(t)}f = D_{\alpha_{p_0}(t),-X_{\alpha_{p_0}(t)}}f = -X_{\alpha_{p_0}(t)}\cdot f \leq 0 \tag{1.48}$$

where we have used [Sch07, Theorem 3.7] and the definition of actions of smooth vector fields on functions. Furthermore, since by definition of being a gradient like vector field, $X \cdot f > 0$ away from the critical points, and $\alpha_{p_0}(t)$ is not constant, we infer that

$$\dot{f}(t) < 0 \qquad \text{for all } t \in \mathbb{R}. \tag{1.49}$$

Now we define the set

$$\Gamma_{\pm\infty} = \{q \in M \mid \exists (t_n)_{n\geq 1} \subset \mathbb{R} : t_n \to \pm\infty, \lim_{n\to\infty}\alpha_{p_0}(t_n) = q\}. \tag{1.50}$$

Since in a compact space, any sequence has a convergent subsequence, we conclude that $\Gamma_{\pm\infty} \neq \emptyset$. What the proposition claims is that this set contains just a single point, which is critical, thus we go on to prove this.

It suffices to look at the case $\Gamma_\infty$, since the other case is completely similar. We claim that

$$\Phi_t(\Gamma_\infty) \subset \Gamma_\infty, \qquad \text{for all } t \geq 0. \tag{1.51}$$

Indeed, suppose $q \in \Gamma_\infty$. Then we have a sequence $(t_n)_{n\geq 0}$ such that $\alpha_{p_0}(t_n) \to q$ and hence

$$\alpha_{p_0}(t_n + t) = \Phi(t_n + t, p_0) = \Phi_t(\alpha_{p_0}(t_n)) \to \Phi_t(q) \in \Gamma_\infty \tag{1.52}$$

and the inclusion follows. Now, suppose $q_0, q_1 \in \Gamma_\infty$. Then by definition, we can find sequences $(t_n^0)_{n\geq 1}$ and $(t_n^1)_{n\geq 0}$ such that $\lim_{n\to\infty}\alpha_{p_0}(t_n^i) = q_i$. We can without loss of generality assume that these are increasing, and defining $(t_n)_{n\geq 1}$ by alternating between these sequences we obtain

$$\alpha_{p_0}(t_{2n+i}) \to q_i, \qquad i = 0, 1, \qquad t_{2n+1} \in (t_{2n}, t_{2n+2}). \tag{1.53}$$

Since by the deduction above we have $\dot{f}(t) < 0$, we conclude that

$$f(\alpha_{p_0}(t_{2n})) > f(\alpha_{p_0}(t_{2n+1})) > f(\alpha_{p_0}(t_{2n+2})). \tag{1.54}$$

Letting $n \to \infty$ we infer by continuity of $f$ that $f(q_0) = f(q_1)$ and thus there exists $c \in \mathbb{R}$ such that

$$\Gamma_\infty \subset f^{-1}(c). \tag{1.55}$$

If $q \in \Gamma_\infty$ is not a critical point, then $\alpha_q(t) \in \Gamma_\infty$ is a non-constant trajectory of $-X$ contained in the level set $f^{-1}(c)$, but this is impossible since by the computations above $f$ is strictly decreasing on such trajectories. We conclude that

$$\Gamma_\infty \subset Cr_f. \tag{1.56}$$

Since critical points of Morse functions are isolated by [Lemma 1.1.1], it suffices to prove that $\Gamma_\infty$ is connected.

Now denote by $\mathcal{C}$ the set of connected components of $\Gamma_\infty$, and assume for contradiction that $|\mathcal{C}| > 1$. Fix a metric $d$ on $M$ and set

$$\delta := \min\{dist(C, C') : C, C' \in \mathcal{C}, C \neq C'\} > 0 \tag{1.57}$$

Let $C_0 \neq C_1 \in \mathcal{C}$ and $q_i \in C_i$ for $i = 0, 1$. Then by the same argument as before, there exists an increasing sequence $(t_n)_{n\geq 1}$ such that

$$\alpha_{p_0}(t_{2n+i}) \to q_i, \qquad i = 0, 1, \qquad t_{2n+1} \in (t_{2n}, t_{2n+2}) \tag{1.58}$$

where we have

$$\lim_{n\to\infty} dist(\alpha_{p_0}(t_{2n}), C_0) = dist(q_0, C_0) = 0 \tag{1.59}$$

$$\lim_{n\to\infty} dist(\alpha_{p_0}(t_{2n+i}), C_0) = dist(q_1, C_0) \geq \delta \tag{1.60}$$

By [Mun00, Mean value theorem] applied to the continuous, real valued function $t \mapsto dist(\alpha_{p_0}(t), C_0)$, we conclude that for $n$ sufficiently large, there exits $s_n \in (t_{2n}, t_{2n+1})$ such that

$$dist(\alpha_{p_0}(s_n), C_0) = \frac{\delta}{2} \tag{1.61}$$

A subsequence of $(\alpha_{p_0}(s_n))_{n\geq 1}$ converges to a point $q \in \Gamma_\infty$ such that $dist(q, C_0) = \frac{\delta}{2}$, but this is impossible since $q \in \Gamma_\infty \subset Cr_f \backslash C_0$. This is the desired contradiction, thus $\Gamma_\infty$ is connected and we conclude that $\Gamma_\infty$ contains exactly one critical point as desired. □

**Definition 1.2.3.** Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold with $p_0$ a critical point of $f$. The *stable manifold* $S(p_0)$ of $p_0$ is the set of points which flow "down" to $p_0$ and the *unstable manifold* $U(p_0)$ of $p_0$ is the set of points which flow "up" to $p_0$, relative to the gradient vector field $X$, that is

$$S(p_0) = S(p_0, X) = \{x \in M \mid \lim_{t\to\infty} \Phi_t(x) = p_0\} \tag{1.62}$$

$$U(p_0) = U(p_0, X) = \{x \in M \mid \lim_{t\to-\infty} \Phi_t(x) = p_0\} \tag{1.63}$$

We furthermore set $F(p_1, p_2) = U(p_1) \cap S(p_2)$. ◁

**Proposition 1.2.1.** *Let $\lambda$ be the index of the critical point $p_0$. Then $S(p_0)$ and $U(p_0)$ are smooth manifolds homeomorphic to $\mathbb{R}^{m-\lambda}$ and $\mathbb{R}^\lambda$ respectively.*

*Proof.* Note that we have $S(p_0, X) = U(p_0, -X)$, hence it suffices to look at the case with the unstable manifold. The proof rests on the following fact that for $\epsilon > 0$ sufficiently small, and $f(p_0) = c_0$, the set $K_\epsilon = U(p_0) \cap f^{-1}(c_0 - \epsilon)$ is a sphere of dimension $\lambda - 1$ smoothly embedded in the level set $f^{-1}(c_0 - \epsilon)$ with trivializable normal bundle. Indeed, by definition of $X$ being a gradient-like vector field, there exists coordinates $(x_i)$ such that $x^(p_0) = 0$ and

$$X = -2\sum_{i=1}^{\lambda} x^i \frac{\partial}{\partial x_i} + 2\sum_{i=\lambda+1}^{m} x^i \frac{\partial}{\partial x^i} \tag{1.64}$$

in the neighborhood $N = \{\sum_{i=1}^{\lambda} x_i^2 + \sum_{i=\lambda+1}^{m} x_i^2\} < r$ for some appropriate $r > 0$. Now an integral curve $\alpha_q(t)$ of $-X$ converging to $p_0$ as $t \to -\infty$ stays inside $N$ for $t$ sufficiently small, but all such integral curves have the form $e^{2t}(x^1, x^2, \ldots, e^\lambda, 0, \ldots, 0)$, cf. [Nic07, p. 35] and they are all included in the $\lambda$-dimensional disc:

$$D(p_0, r) = \Big\{ \sum_{i=1}^{\lambda} x_i^2 \leq r, x_i = 0 \text{ for } i > \lambda \Big\} \tag{1.65}$$

By the arguments from [Lemma 1.2.3], $f$ is strictly decreasing on such non-constant trajectories, thus if $\epsilon < r$, then

$$K_\epsilon = U(p_0) \cap f^{-1}(c_0 - \epsilon) = \partial D(p_0, \epsilon) \overset{\varphi}{\simeq} S^{\lambda-1} \tag{1.66}$$

which was the claim. Finally, let $(r, \theta) \in \mathbb{R} \times S^{\lambda-1}$ denote the polar coordinates on $\mathbb{R}^\lambda$ and define the following smooth map

$$F : \mathbb{R}^\lambda \to U(p_0) \qquad \text{by} \qquad F(r, \theta) = \alpha_{\varphi^{-1}(\theta)}(\tfrac{1}{2} \log r) \tag{1.67}$$

Which by the above derivations is seen to be the required diffeomoprhism. $\square$

It follows that a Morse function $f$ on $M$ provides two decompositions of $M$ into disjoint cells:

$$M = \bigcup_p S(p) = \bigcup_p U(p) \tag{1.68}$$

where the union is over all critical points $p$ of $f$. These will be called the stable and unstable manifold decompositions respectively. We note that the Morse condition is of crucial importance here. Indeed, in general, critical points are not necessarily isolated, and flow lines do not necessarily converge to critical points, cf. [Gue02]. However, given the unstable manifold decomposition above, we wish to examine how these cells fit together, and this will be the theme of the next section.

## 1.3 The Morse Principles

This section will contain the main theorems of classical Morse Theory. In the following we assume that $M$ is a smooth, $m$-dimensional manifold, and that $f : M \to \mathbb{R}$ is an *exhaustive* Morse function, that is, the sublevel set:

$$M^t = f^{-1}(-\infty, t] = \{p \in M \mid f(p) \leq t\} \tag{1.69}$$

is compact for every $t \in \mathbb{R}$. We will explain how these sublevel sets in a certain sense stays unchanged when we do not meet any critical points, and, more surprisingly, how a change looks when we do pass a critical point of some index $\lambda$. This is the weak and strong principles of Morse theory respectively, which we will now prove, following [Nic07] and [Mil68] respectively.

### 1.3.1 The Weak Morse Principle

**Theorem 1.3.1** (Weak Morse principle)**.** *Suppose that the set $f^{-1}[a, b]$ is compact and contains no critical points of $f$. Then $M^a$ is diffeomorphic to $M^b$. Furthermore $M^a$ is a deformation retract of $M^b$, so that the inclusion $M^a \hookrightarrow M^b$ is a homotopy equivalence.*

*Proof.* We clearly have $f^{-1}[a, b] \subset M^t$ for $t > b$, which by assumption is compact, thus $M^c$ contains only finitely many critical points, cf. [Corollary 1.1.2]. Since we have assumed that $f^{-1}[a, b]$ has no critical points, we infer that there exists $\epsilon > 0$ such that $f^{-1}[a - \epsilon, b + \epsilon]$ has no critical points.

Fix a gradient-like vector field $Y$ and construct a smooth function $\rho : M \to [0, \infty)$ by

$$\rho(p) = \begin{cases} |Y_p f|^{-1} & \text{if} \quad f(p) \in [a, b] \\ 0 & \text{if} \quad f(p) \notin [a - \epsilon, b + \epsilon] \end{cases} \tag{1.70}$$

using suitable bump functions. Note that this is well defined since $f^{-1}[a, b]$ contains no critical points by assumption. Define a new vector field $X$ on $M$ by $X = -\rho Y$. Then $X$ satisfies the conditions of [Lemma 1.2.1] hence it generates a flow $\Phi$ on $M$, such that for an integral curve $\alpha_p(t) : \mathbb{R} \to M$

defined by $t \mapsto \Phi_t(p)$, see [BJ82, Definition 8.3], we have $\frac{d}{dt}\alpha_p(t) = X_{\alpha_p(t)}$. Differentiating along this curve we see that in the region $f^{-1}[a,b]$, we have

$$\frac{d}{dt}f(\alpha_p(t)) = X_{\alpha_p(t)}f = -\rho(\alpha_p(t))Y_{\alpha_p(t)}f = \frac{-1}{Y_{\alpha_p(t)}f}Y_{\alpha_p(t)}f = -1 \tag{1.71}$$

hence in the region $f^{-1}[a,b]$, the function $f$ decreases at constant unit speed. Therefore, for $p \in M^b$ we have $f(\Phi_{b-a}(p)) = f(\alpha_p(b-a)) \leq f(b-(b-a)) = f(a)$ and similarly for $p \in M^a$, $f(\Phi_{a-b}(p)) \leq f(b)$. We conclude that

$$\Phi_{b-a}(M^b) = M^a \qquad \text{and} \qquad \Phi_{a-b}(M^a) = M^b \tag{1.72}$$

hence $\Phi_{b-a}$ is a diffeomorphism between $M^b$ and $M^a$ as required. To see that $M^a$ is a deformation retract, define the homotopy

$$H : M^b \times [0,1] \to M^b \qquad \text{by} \qquad H(p,t) = \begin{cases} p & \text{if} \quad f(p) \leq a \\ \Phi_{t(f(p)-a)}(p) & \text{if} \quad a \leq f(p) \leq b \end{cases} \tag{1.73}$$

Then for $p \in M^a$ we have $H(t,p) = p$ for all $t \in [0,1]$ while for $p \in M^b$ we have $H(p,x) = \Phi_{f(p)-a}(p) \in M^a$, that is $H$ is a homotopy between the identity and a retraction i.e. $H$ is a deformation retraction of $M^b$ onto $M^a$ as desired, cf. [Hat02]. □

### 1.3.2 Strong Morse Principle

The problem is therefore to determine how the shape of $M^t$ changes as the parameter $t$ passes through a critical value. Surprisingly, the information of the index of the critical point is sufficient to show the following beautiful result, outlined in [Mil68].

**Theorem 1.3.2** (Strong Morse Principle)**.** *Let $p_0$ be a non-degenerate critical point with index $\lambda$. Setting $f(p_0) = c$, suppose that $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical point for some $\epsilon > 0$. Then for all $\epsilon$ sufficiently small, the set $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon}$ with a $\lambda$-cell attached.*

*Proof.* By [Morse Lemma], we can find local coordinates $(u_i)$ in a neighborhood $U$ of $p$ such that

$$f = c - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_m^2 \tag{1.74}$$

and $u_i(p_0) = 0$ for all $i$. Choose $\epsilon > 0$ sufficiently small such that

- $f^{-1}[c-\epsilon, c+\epsilon]$ is compact and contains no other critical points than $p$.
- The image of $U$ under the diffeomorphism $(u_1, \dots, u_m) : U \to \mathbb{R}^m$ contains the closed disc

$$D = \{(u_1, \dots, u_m) \mid \sum_{i=1}^m u_i^2 \leq 2\epsilon\} \tag{1.75}$$

Define the $\lambda$-cell, $e^\lambda$, to be the subset of $U$ with

$$u_1^2 + \cdots + u_\lambda^2 \leq \epsilon \qquad \text{and} \qquad u_{\lambda+1} = \cdots = u_m = 0 \tag{1.76}$$

Then we are in the situation depicted in the following figure, in the case of $f$ being the height function and $M$ the torus:

Note that it makes sense to attach such a cell to $M^{c-\epsilon}$, since their intersection satisfies:

$$\begin{aligned} e^\lambda \cap M^{c-\epsilon} &= \{\mathbf{u} \mid \sum_{i=1}^\lambda u_i^2 \leq \epsilon\} \cap \{x \in M \mid f(x) \leq c - \epsilon\} = \{\mathbf{u} \mid \sum_{i=1}^\lambda u_i^2 \leq \epsilon, -\sum_{i=1}^\lambda + \sum_{i=\lambda+1}^m u_i^2 \leq -\epsilon\} \\ &= \{\mathbf{u} \mid \sum_{i=1}^\lambda u_i^2 = \epsilon, u_{\lambda+1} = \cdots = u_m = 0\} = \partial e^\lambda \end{aligned} \tag{1.77}$$

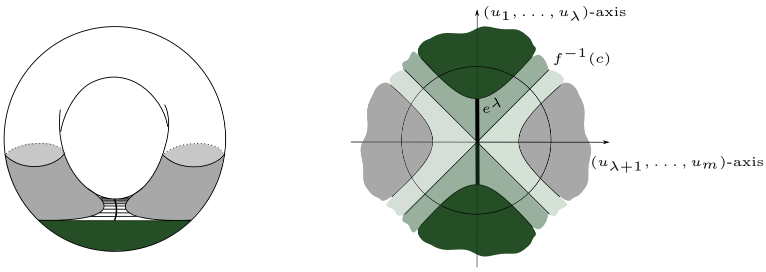
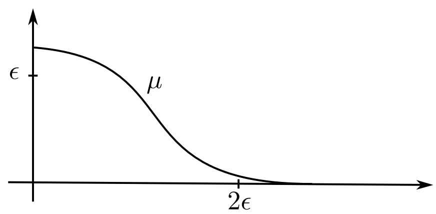



Figure 1.5: Schematic presentation, when $f$ is the height function on $T^2$.

We must prove that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c+\epsilon}$. We will do this in two steps. For the first step, we will construct a new function $F : M \to \mathbb{R}$ which coincides with the original Morse function $f$ except in a neighborhood, where $F < f$, allowing us to utilize the weak Morse principle and thus deformation retract $M^{c+\epsilon}$ to a sublevelset of $F$ still containing $e^\lambda$. In the second step, we will then find an explicit deformation retraction of this sublevelset onto $M^{c-\epsilon} \cup e^\lambda$.

In order to define the function $F$, we first define the smooth function $\mu : \mathbb{R} \to \mathbb{R}$ such that

$$\mu(0) > \epsilon, \qquad \mu(r) = 0 \text{ for } r > 2\epsilon, \qquad -1 < \mu'(r) \leq 0 \text{ for all } r \tag{1.78}$$




Now let $F : M \to \mathbb{R}$ coincide with $f$ outside the neighborhood $U$, and let

$$F = f - \mu\Big(\sum_{i=1}^{\lambda} u_i^2 + 2\sum_{i=\lambda+1}^{m} u_i^2\Big) \tag{1.79}$$

within $U$. Then $F$ is well defined and smooth on $M$. Indeed, since $f$ is smooth, $F$ is a composition of smooth functions inside $U$ and obviously outside $U$. Furthermore, since $\mu$ is zero outside $D$, which is contained in $U$ by construction, we conclude that $F$ is smooth.

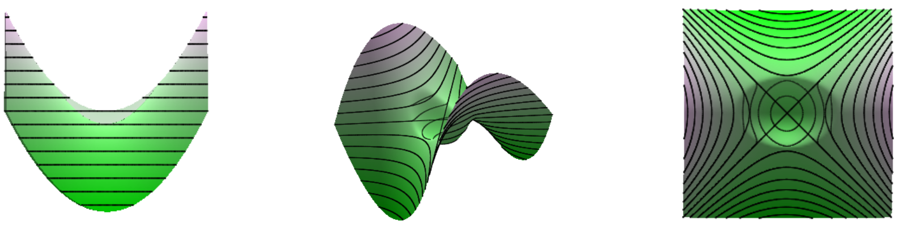


Figure 1.6: The functions $F$ and $f$

Define yet another two functions

$$\zeta, \eta : U \to [0, \infty) \qquad \text{by} \qquad \zeta = \sum_{i=1}^{\lambda} u_i^2 \qquad \text{and} \qquad \eta = \sum_{i=\lambda+1}^{m} u_i^2 \tag{1.80}$$

Then clearly, in the neighborhood $U$, we have $f = c - \zeta + \eta$ and thus

$$F(p) = c - \zeta(p) + \eta(p) - \mu(\zeta(p) + 2\eta(p)) \tag{1.81}$$

for all $p \in U$. Next we need the following intermediate lemma:

**Lemma 1.3.1.** *The function $F$ satisfies the following properties:*

*(i) $F$ is a Morse function with the same critical points as $f$.*

*(ii) The region $F^{-1}(-\infty, c+\delta]$ coincides with the region $f^{-1}(-\infty, c+\delta]$ for all $\delta \geq \epsilon$.*

*Proof.* Clearly outside $D$, the function $F$ and $f$ have the same critical points since they are identical in this region. Now inside $D$ we have

$$F = f - \mu(\zeta + 2\eta) = c - \zeta + \eta - \mu(\zeta + 2\eta). \tag{1.82}$$

Therefore we obtain the following, using the defining property of the derivative of $\mu$:

$$\frac{\partial F}{\partial \zeta} = -1 - \mu'(\zeta + 2\eta) < 0, \qquad \text{and} \qquad \frac{\partial F}{\partial \eta} = 1 - 2\mu'(\zeta + 2\eta) \geq 1 \tag{1.83}$$

Furthermore we see that

$$dF = \frac{\partial F}{\partial \zeta} d\zeta + \frac{\partial F}{\partial \eta} d\eta, \tag{1.84}$$

where $d\zeta$ and $d\eta$ are only zero at the origin. Since the other factors are bounded away from zero, we conclude that $F$ has no other critical points in $U$, and the first part follows.

To see that (ii) holds, note that we clearly have $F^{-1}(-\infty, c+\delta] \cap D^c = f^{-1}(-\infty, c+\delta] \cap D^c$. Furthermore, since $F \leq f$, we have the general inclusion

$$f^{-1}(-\infty, a] = \{x \in M \mid f(x) \leq a\} \subset \{x \in M \mid F(x) \leq a\} = F^{-1}(-\infty, a] \tag{1.85}$$

for all $a \in \mathbb{R}$. It thus suffices to prove the inclusion $F^{-1}(-\infty, c+\delta] \cap D \subset f^{-1}(-\infty, c+\delta] \cap D$. Suppose $p \in F^{-1}(-\infty, c+\delta] \cap D$. Then $\zeta(p) + \eta(p) \leq 2\epsilon$ and $F(p) = c - \zeta(p) + \eta(p) - \mu(\zeta(p) + 2\eta(p)) \leq c + \delta$ which gives us

$$\eta(p) \leq \zeta(p) + \delta + \mu(\eta(p) + 2\zeta(p)) \tag{1.86}$$

Using the fact that $-1 < \mu'(r) \leq 0$ we get $\mu(t) - \mu(2\epsilon) \leq 2\epsilon - t$ for all $t \leq 2\epsilon$ and hence

$$\mu(t) = \mu(t) - \mu(2\epsilon) \leq 2\epsilon - t \leq 2\delta - t \tag{1.87}$$

for all $t \leq 2\epsilon$, where we use that $\mu(2\epsilon) = 0$. Therefore, if $\zeta + 2\eta \leq 2\epsilon$, we obtain

$$\eta(p) \leq \zeta(p) + \delta + \mu(\eta(p) + 2\zeta(p)) \leq \zeta(p) + \delta + 2\delta - \zeta(p) - 2\eta(p) = 3\delta - 2\eta(p) \tag{1.88}$$

and thus $\eta(p) \leq \delta$. This in turn gives $\eta(p) - \zeta(p) \leq \delta$ i.e. $f - c \leq \delta$ and thus finally $f \leq c + \delta$ as desired. Clearly, if $\zeta(p) + 2\eta(p) \geq 2\epsilon$ then $f(p) = F(p) \leq c + \epsilon$, completing the proof. ☐

We can now proceed with the proof of the strong Morse principle. Consider $F^{-1}[c-\epsilon, c+\epsilon]$. By the lemma and the inclusion $F^{-1}[c-\epsilon, \infty) \subset f^{-1}[c-\epsilon, \infty)$, which follows by the same argument as in (1.85), we see that

$$\begin{aligned} F^{-1}[c-\epsilon, c+\epsilon] = F^{-1}((-\infty, c+\epsilon] \cap [c-\epsilon, \infty)) = F^{-1}(-\infty, c+\epsilon] \cap F^{-1}[c-\epsilon, \infty) \\ \subset f^{-1}(-\infty, c+\epsilon] \cap f^{-1}[c-\epsilon, \infty) = f^{-1}[c-\epsilon, c+\epsilon] \end{aligned} \tag{1.89}$$

Furthermore, since $f^{-1}[c-\epsilon, c+\epsilon]$ is assumed compact, $F^{-1}[c-\epsilon, c+\epsilon]$, being a closed subset, is itself compact. Since $p$ is the only critical point in $f^{-1}[c-\epsilon, c+\epsilon]$, and $F$ has the same critical points, $p$ is the only possible critical point in $F^{-1}[c-\epsilon, c+\epsilon]$, but

$$F(p) = c - \zeta(p) + \eta(p) - \mu(\zeta(p) + 2\eta(p)) = c - \mu(0) < c - \epsilon \tag{1.90}$$

thus $F^{-1}[c-\epsilon, c+\epsilon]$ contains no critical points. Therefore, by [Weak Morse Principle] and the lemma we see that $F^{-1}(-\infty, c-\epsilon]$ is a deformation retract of $F^{-1}(-\infty, c+\epsilon] = f^{-1}(-\infty, c+\epsilon] = M^{c+\epsilon}$. This was the first step.

Now, for the second step, denote $F^{-1}(-\infty, c-\epsilon]$ by $M^{c-\epsilon} \cup H$ where $H$ is the closure of $F^{-1}(-\infty, c-\epsilon] - M^{c-\epsilon}$, and let the cell $e^\lambda$ be the set of points $p \in U$ with $\zeta(p) \leq \epsilon$, $\eta(p) = 0$. Note that $e^\lambda \in H$. Indeed, since [calculation to come]

We claim that $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H$. This is depicted schematically in figure 1.7. More explicitly, we define a deformation retraction $R : (M^{c-\epsilon} \cup H) \times I \to M^{c-\epsilon}$ as follows:

$$R(p,t) = \begin{cases} p & \text{if} \quad p \notin U \\ (u_1, \dots, u_\lambda, tu_{\lambda+1}, \dots, tu_m) & \text{if} \quad p \in U, \zeta \le \epsilon \\ (u_1, \dots, u_\lambda, s(t)u_{\lambda+1}, \dots, s(t)u_m) & \text{if} \quad p \in U, \epsilon \le \zeta \le \eta + \epsilon \\ (u_1, \dots, u_m) & \text{if} \quad p \in U, \eta + \epsilon \le \zeta \end{cases} \tag{1.91}$$

where $s : [0,1] \to \mathbb{R}$ is given by $s(t) = t + (1-t)\sqrt{(\zeta - \epsilon)/\eta}$. We must check that this is indeed a well defined deformation retraction, and divide this into the three nontrivial interesting cases, see figure 1.7.

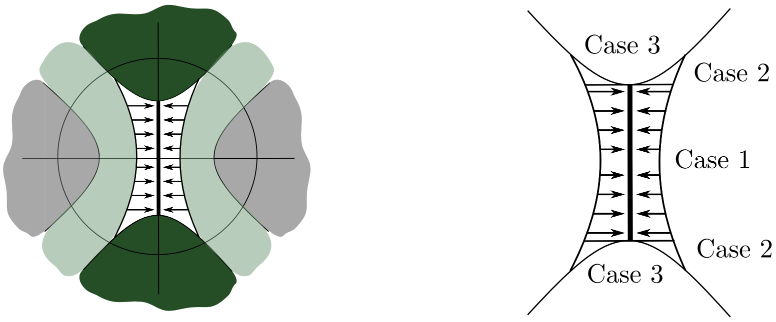



Figure 1.7: Schematic presentation

1. For $\zeta \le \epsilon$ we have $R(p,1) = (u_1, \dots, u_m) = p$ and $R(p,0) = (u_1, \dots, u_\lambda, 0, \dots, 0) \in e^\lambda$. Thus in this region $R$ is a homotopy between the identity and a retraction, which collapses the entire region into $e^\lambda$, i.e. $R$ is a deformation retraction.

2. For $\epsilon \le \zeta \le \eta + \epsilon$ we first note, that the function $s$ is well defined and continuous since by assumption $\zeta - \epsilon \ge 0$. Since $s(1) = 1$ and $s(0) = \sqrt{(\zeta - \epsilon)/\eta}$ we have $R(p,1) = (u_1, \dots, u_m) = p$ and $R(p,0) = (u_1, \dots, u_\lambda, \sqrt{(\zeta-\epsilon)/\eta}\, u_{\lambda+1}, \dots, \sqrt{(\zeta-\epsilon)/\eta}\, u_m)$ and since

$$\begin{aligned} &f(u_1, \dots, u_\lambda, \sqrt{(\zeta-\epsilon)/\eta}\, u_{\lambda+1}, \dots, \sqrt{(\zeta-\epsilon)/\eta}\, u_m) \\ &= c - u_1^2 - \dots - u_\lambda^2 + \frac{\zeta - \epsilon}{\eta} u_{\lambda+1}^2 + \dots + \frac{\zeta - \epsilon}{\eta} u_m^2 \\ &= c - \zeta + \frac{\zeta - \epsilon}{\eta} \eta = c - \epsilon \end{aligned} \tag{1.92}$$

we have $R(p,0) \in f^{-1}(c - \epsilon)$. Thus again, we have a deformation retraction. Note that this corresponds to case 1 since when $\zeta = \epsilon$ we have $s(t) = t$.

3. Finally, if $\eta + \epsilon \le \zeta$, or in other words $f = c - \zeta + \eta \le c - \epsilon$, that is we are in $M^{c-\epsilon}$, then $R(p,t) = (u_1, \dots, u_m)$. This coincides with the previous case, since when $\zeta = \eta + \epsilon$ we have $s(t) = t + (1-t)\sqrt{(\zeta - \epsilon)/\eta} = t + (1-t)\sqrt{\eta/\eta} = 1$.

We conclude that $R$ is indeed a well defined deformation retraction and thus $M^{c-\epsilon} \cup e^\lambda$ is a deformation retract of $M^{c-\epsilon} \cup H = F^{-1}(-\infty, c - \epsilon]$ which by the above is again a deformation retract of $M^{c+\epsilon}$. Composing these gives the desired homotopy equivalence. □

We can generalize this to the general case where we have $k$ non-degenerate critical points in the pre-image, using the fact that these are isolated, cf. [Corollary 1.1.1]. The proof rests on the fact that we can raise and lower critical values without changing the critical points, see [Mat02]. We state and prove this fact for future reference:

**Lemma 1.3.2** (Raising and lowering critical values)**.** *Suppose $f : M \to \mathbb{R}$ is a Morse function with critical points $p_1, \dots, p_k$. Then there exists a Morse function $f' : M \to \mathbb{R}$ with the same critical points where $f'(p_i) \neq f'(p_j)$ for $i \neq j$.*

*Proof.* Suppose $f(p_1) = f(p_2) = c$. By [Morse lemma] we can find local coordinates $(u_i)$ such that

$$f = c - u_1^2 - \cdots - u_\lambda^2 + u_{\lambda+1}^2 + \cdots + u_m^2 \tag{1.93}$$

Let $X$ be a gradient-like vector field for $f$. Then we have

$$X \cdot f = \sum_{i=1}^{m} \Big(\frac{\partial f}{\partial u_i}\Big) = 4\sum_{i=1}^{m} u_i \tag{1.94}$$

Since the critical points are isolated by [Corollary 1.1.1], we can find $\epsilon > 0$ such that the $m$-dimensional discs $\overline{B}(p_1, \epsilon)$ and $B(p_1, 2\epsilon)$ does not contain any other critical points. Furthermore, in the region $\overline{B}(p_1, 2\epsilon) - B(p_1, \epsilon)$ we have

$$4\epsilon^2 \leq X \cdot f \leq 4(2\epsilon)^2 \tag{1.95}$$

There exists a bump function

$$h \in C^\infty(M) \qquad \text{such that} \qquad 0 \leq h \leq 1$$
$$\text{and} \qquad h(q) = 1 \quad \text{for} \quad q \in B(p_1, \epsilon) \qquad \text{and} \qquad h(q) = 0 \quad \text{for} q \notin B(p_1, 2\epsilon)$$

see for example [Sch07, Lemma 5.4]. Now we define the function

$$\tilde{f} : M \to \mathbb{R} \qquad \text{by} \qquad \tilde{f} = f + ah \tag{1.96}$$

for some $a \in \mathbb{R}\backslash 0$ small enough. Outside $B(p_1, 2\epsilon)$ we have $f = \tilde{f}$ and in $B(p_1, \epsilon)$ we still only have $p_1$ as critical point since $h = 1$ in this region. Therefore, we only need to check that ??? contains no critical points. To see this, note that

$$\Big|\frac{\partial f}{\partial u_i} - \frac{\partial \tilde{f}}{\partial u_i}\Big| = \Big|a\frac{\partial f}{\partial u_i}\Big|, \qquad i = 1, \ldots, m \tag{1.97}$$

Then by continuity, we conclude that the distance

$$\Big|\sum_{i=1}^{m} \Big(\frac{\partial f}{\partial u_i}\Big)^2 - \sum_{i=1}^{m} \Big(\frac{\partial \tilde{f}}{\partial u_i}\Big)^2\Big| \tag{1.98}$$

can be made as small as desired. In particular, since $X \cdot f$ takes the minimum value $4\epsilon^2 > 0$ in $\sum_{i=1}^{m} \Big(\frac{\partial f}{\partial u_i}\Big)^2$, we can make $\sum_{i=1}^{m} \Big(\frac{\partial \tilde{f}}{\partial u_i}\Big)^2$ attain a positive minimum value by choosing $a$ small enough. Therefore, $\tilde{f}$ has the same set of critical values as $f$, and by construction they are still non-degenerate. We conclude that $\tilde{f}$ is a Morse function, and

$$\tilde{f}(p_1) = f(p_1) + a, \qquad \tilde{f}(p_2) = f(p_2) \tag{1.99}$$

thus $\tilde{f}(p_1) \neq \tilde{f}(p_2)$. Continuing in this fashion gives the required function. □

From this fact, the general case reduces to what we have already proved, thus we immediately arrive at the following corollary, which we will likewise call the strong Morse principle.

**Corollary 1.3.1.** *Suppose $f^{-1}(c)$ contains $k$ non-degenerate critical points $p_1, \ldots, p_k$ with indices $\lambda_1, \ldots, \lambda_k$. Then $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$.*

### 1.3.3 CW decomposition

With [Corollary 1.3.1], also called the structural theorem of classical Morse theory, now in place, we will here prove the Main Theorem, namely that we can give a manifold with an exhaustive Morse function defined on it a CW-complex structure, again following the beautiful exposition of [Mil68].

**Theorem 1.3.3** (Main Theorem)**.** *If $f : M \to \mathbb{R}$ is an exhaustive Morse function, then $M$ has the homotopy type of a CW-complex, with one cell of dimension $\lambda$ for each critical point of index $\lambda$.*

*Proof.* Let $c_1 < c_2 < \dots$ be the critical values of $f$, of which there might be infinitely many. Since $f$ is exhaustive, each $M^a$ is compact, and thus, by the proof of 1.1.2, the sequence $(c_i)_{i\geq 1}$ can have no limit point. We can therefore prove the theorem by induction. Clearly, $M^a = \emptyset$ for $a < c_1$, thus the induction start is vacuously satisfied. Now suppose $a \neq c_1, c_2, \dots$ and that $M^a$ has the homotopy type of a CW-complex, that is there exits a homotopy equivalence $g : M^a \to K$ where $K$ is a CW-complex. Let $c$ be the smallest $c_i > a$ and $j(c) = \sharp f^{-1}(c)$. Then for $\epsilon > 0$ small enough, it follows by [Corollary 1.3.1] that $M^{c+\epsilon}$ has the homotopy type of $M^{c-\epsilon} \cup e^{\lambda_1} \cup_{\varphi_i} \cdots \cup_{\varphi_{j(c)}} e^{\lambda_{j(c)}}$ for certain attaching maps $\varphi_1, \dots, \varphi_{j(c)}$. Furthermore, by [Weak Morse Principle], there is a homotopy equivalence $h : M^{c-\epsilon} \to M^a$.

Then by [Hat02, Cellular Approximation], each map $g \circ h \circ \varphi_i : \partial e^{\lambda_i} \to K$, going between CW-complexes is homotopic to a cellular map, i.e. a map which maps cells to cells of the same or lower dimension. Therefore $g \circ h \circ \varphi_i$ is homotopic to a map

$$\psi_i : \partial e^{\lambda_i} \to K^{\lambda - 1} \tag{1.100}$$

where $K^{\lambda-1}$ denotes the $(\lambda - 1)$-skeleton of the CW complex $K$. Since the map $g \circ h : M^{c-\epsilon} \to K$ is a homotopy equivalence, it extends to a homotopy equivalence [Elaborate, do we know lemma 3.7 in Milnor?]

$$F : M^{c-\epsilon} \cup_{\varphi_1} e^{\lambda_1} \cup_{\varphi_2} \cdots \cup_{\varphi_j(c)} e^{\lambda_{j(c)}} \to K \cup_{\psi_1} e^{\lambda_1} \cup_{\psi_2} \cdots \cup_{\psi_{j(c)}} e^{\lambda_{j(c)}} \tag{1.101}$$

where $K \cup_{\psi_1} e^{\lambda_1} \cup_{\psi_2} \cdots \cup_{\psi_{j(c)}} e^{\lambda_{j(c)}}$ is a CW-complex of the same homotopy type as $M^{c+\epsilon}$ by the above. By induction, it follows that each $M^t$ has the homotopy type of a CW-complex.

If $M$ is compact, then $M = M^t$ for $t = \max\{f(x) \mid \mathbf{x} \in M\}$ and we are done. If $M$ is not compact, but all critical points lie in some $M^t$, then by a slight modification of [Weak Morse Principle], we see that $M$ deformation retracts to $M^t$ and thus we are done. Finally, if there are infinitely many critical points, we obtain, by the above construction, a whole sequence of homotopy equivalences between the sublevel sets $M^{a_i}$ and corresponding CW-complexes $K_i$ as follows:

$$\begin{tikzcd} M^{a_1} \arrow[r, hook] \arrow[d] & M^{a_2} \arrow[r, hook] \arrow[d] & M^{a_3} \arrow[r, hook] \arrow[d] & \cdots \\ K_1 \arrow[r, hook] & K_2 \arrow[r, hook] & K_3 \arrow[r, hook] & \cdots \end{tikzcd}$$

each extending the previous one. Now, let $K = \bigcup_i K_i$ in the weak topology, see [Fol99, p. 120], and let $l : M \to K$ be the limit map. Then $l$ induces isomorphisms of homotopy groups on all dimensions hence by [Hat02, Whitehead's Theorem], $l$ is a homotopy equivalence as required. ☐

### 1.3.4 Examples and applications

With the main theorem of classical Morse theory now at our disposal, we pause the development of the theory for a moment, to look at some interesting examples and immediate applications, inspired by [Mil68]. We shall also touch upon what will turn out to be our first premature example of a complex Grassmannian, the complex projective space.

**Proposition 1.3.1.** *Any smooth manifold has the homotopy type of a CW-complex.*

*Proof.* By the Existence theorem of Morse functions, any manifold $M$ has a Morse function for which each $M^a$ is compact. Now the main theorem gives the required CW-decomposition of $M$. ☐

**Proposition 1.3.2** (Reeb)**.** *If $f : M \to \mathbb{R}$ is a Morse function on a compact manifold with only two critical points, then $M$ is homeomorphic to a sphere.*

*Proof.* Since any continuous function on a compact set has a minimum and a maximum, the two critical points must correspond to these. Say $f(p_-) = 0$ is the minimum and $f(p_+)$ is the maximum. By [Morse lemma], we know that in a neighborhood of $p_-$ we can write $f$ in the standard form. Since we cannot attain smaller values, the index $\lambda$ has to be 0 and therefore, for $\epsilon > 0$ small enough:

$$M^\epsilon = f^{-1}[0, \epsilon] = \{\mathbf{x} \in M \mid f(\mathbf{x}) \leq \epsilon\} = \{\mathbf{x} \in M \mid x_1^2 + \cdots + x_m^2 \leq \epsilon\} \simeq D^m$$

Likewise for the point $p_+$ corresponding to the maximal value, we cannot attain larger values hence the index is $m$ and therefore we obtain:

$$f^{-1}[1-\epsilon, 1] = \{\mathbf{x} \in M \mid f(\mathbf{x}) \geq \epsilon\} = \{\mathbf{x} \in M \mid 1 - x_1^2 - \cdots - x_m^2 \geq 1 - \epsilon\} \simeq D^m$$

Furthermore, since $f^{-1}[\epsilon, 1-\epsilon]$ is compact, being a closed subset of the compact space $M$, and it contains no critical points, it follows from [Theorem 1.3.1, Weak Morse Principle] that $M^\epsilon$ is diffeomorphic to $M^{1-\epsilon}$. Since $M = M^{1-\epsilon} \cup f^{-1}[1-\epsilon, 1]$, we conclude that $M$ is the union of two closed $n$-discs, which are clearly homeomorphic to the northern and southern hemisphere, glued along their common boundary, that is $M$ is homeomorphic to $S^m$. $\square$

**Note 3.** If we furthermore have $\dim M \leq 6$ in the theorem above, then it turns out that $M$ is actually *diffeomorphic* to a sphere. That this is not the case in general follows from the highly non-trivial fact, that a 7-sphere can have several differentiable structures, see [Mil56]. $\diamondsuit$

**Example 1.3.1.** We shall here derive a CW-complex structure of $\mathbb{C}P^n$, which we will later see is the Grassmannian $Gr_1(\mathbb{C}^n)$. It is well known that $\mathbb{C}P^n$, the set of 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$ with the quotient topology inherited from the natural projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to \mathbb{C}P^n$, which we will think of as the equivalence classes of $(n+1)$-tuples $(z_0, \ldots, z_n)$ of complex numbers with $\sum_{j=0}^n |z_j|^2 = 1$, is a smooth, compact manifold, when we equip it with the atlas:

$$U_i = \{(z_0 : z_1 : \cdots : z_n) \mid z_i \neq 0\}, \tag{1.102}$$

$$\varphi_i : U_i \to \mathbb{C}^n \quad \text{defined by} \quad \varphi_i(z_0 : \cdots : z_n) = |z_i|\Big(\frac{z_0}{z_i}, \ldots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \ldots, \frac{z^{n+1}}{z_i}\Big). \tag{1.103}$$

Now define the real function $f : \mathbb{C}P^n \to \mathbb{R}$ by

$$f(z_0 : z_1 : \cdots : z_n) = \sum_{i=0}^n c_i |z_i|^n \tag{1.104}$$

where $c_i \in \mathbb{R}$ are fixed and $c_i \neq c_j$ for $i \neq j$. We claim that this is a Morse function. Indeed, if we divide the complex coordinates in the mage of $\varphi_i$ into real and imaginary parts, we have

$$|z_i|\frac{z_j}{z_i} = x_j + iy_j \tag{1.105}$$

where we have suppressed the $i$. Now, if we just consider the case $i = 0$, then $x_1, y_1, \ldots, x_n, y_n : U_0 \to \mathbb{R}$ maps $U_0$ diffeomorphically onto the open unit ball in $\mathbb{R}^{2n}$. We have

$$x_j^2 + y_j^2 = (x_j + iy_j)(x_j - iy_j) = |z_i|\frac{z_j}{z_i}|z_i|\frac{\overline{z}_j}{\overline{z}_i} = |z_j|^2 \qquad \text{and thus} \tag{1.106}$$

$$|z_0|^2 = \sum_{j=1}^n |z_j|^2 = \sum_{j=0}^n |z_j|^2 - \sum_{j=1}^n |z_j|^2 = 1 - \sum_{j=1}^n |z_j|^2 = 1 - \sum_{j=1}^n (x_j + y_j) \tag{1.107}$$

Therefore, in the coordinate neighborhood $U_0$, we can write the function $f$ as:

$$f = c_0|z_0|^2 + \sum_{i=1}^n c_i|z_i|^n = c_0(1 - \sum_{j=1}^n (x_j + y_j)) + \sum_{j=1}^n c_j(x_j + y_j) = c_0 + \sum_{j=1}^n (c_j - c_0)(x_j + y_j)$$

We conclude that the only critical point of $f$ in $U_0$ is the point $p_0 = (1 : 0 : \cdots : 0)$, which is furthermore non-degenerate. The index is $2\delta_0$ where $\delta_0 = \sharp\{j \mid c_j < c_0, 0 \leq j \leq n\}$. Similarly, we have critical points:

$$p_1 = (0 : 1 : 0 : \cdots : 0), \ldots, p_n = (0 : \cdots : 0 : 1) \tag{1.108}$$

which are then the only critical points of $f$, and the index of $p_k$ is $2\delta_k$ where $\delta_k = \sharp\{j \mid c_j < c_k, 0 \leq j \leq n\}$. Therefore, every even integer between 0 and $2n$ occurs precisely once. By the main theorem, $\mathbb{C}P^n$ has the homotopy type of a CW-complex of the form

$$e^0 \cup e^2 \cup \cdots \cup e^{2n} \tag{1.109}$$

This CW-structure contains an immense amount of information, and thus shows the strength of the main theorem. For example, by cellular homology, one can immediately see that $H_i(\mathbb{C}P^n) = \mathbb{Z}$ for $i$ even and $H_i(\mathbb{C}P^n) = 0$ for $i$ odd. In the final chapter, we shall engage in the project of generalizing results as these to a more general class of spaces – the Grassmannians.

## 1.4 Morse inequalities

As the previous sections clearly show, there is a subtle relation between the topology of $M$ and the critical point data of a function $f : M \to \mathbb{R}$. Although the main theorem does not tell us anything about the attaching maps, which is a serious disadvantage, we can still deduce quite a lot of information about the topology of $M$. The main example illustrating this is the so called *Morse inequalities.* These states that if $f$ is a Morse function on a compact manifold, $M$, then the number of critical points of index $\lambda$ is greater then or equal to the $\lambda$-th Betti number of $M$, [Gue02]. In other words, topology provides a constraint on analysis.

### 1.4.1 Terminology

We will first need to introduce some terminology, following [Nic07], which, as we shall see, will soon turn out to be a reasonable investment for the coming sections as well. Denote by $\mathbb{Z}[t, t^{-1}]$ the ring of formal Laurent series with integer coefficients, that is

$$\mathbb{Z}[t, t^{-1}] = \Big\{ \sum_{n\in\mathbb{Z}} a_n t^n \,\Big|\, a_n \in \mathbb{Z}, a_n = 0\, \forall n \ll 0 \Big\} \tag{1.110}$$

and define an order relation $\succ$ on this ring as follows:

$$X(t) \succ Y(t) \Leftrightarrow \exists Q \in \mathbb{Z}[t, t^{-1}] \text{ with nonnegative coefficients such that } X(t) = Y(t) + (1 + t)Q(t)$$

**Note 4.** The (partial) order relation $\succ$ is well defined. Indeed, it is reflexive since $X \succ X$ with $Q \equiv 0$. It is antisymmetric since if $X \succ Y$ and $Y \succ X$, then $X(t) = Y(t) + (1 + t)Q_1(t)$ and $Y(t) = X(t)+(1+t)Q_2(t)$ and thus $X(t) = X(t)+(1+t)(Q_1(t)+Q_2(t))$ hence $Q_1 = Q_2 \equiv 0$, that is $X = Y$. Finally, it is transitive since if $X \succ Y$ and $Y \succ Z$ then $X(t) = Z(t)+(1+t)(Q_1(t)+Q_2(t))$ with obvious notation, where $Q_1 + Q_2 \in \mathbb{Z}[t, t^{-1}]$ has nonnegative coefficients, thus $X \succ Z$ as required. We conclude that $\succ$ is indeed a partial order relation. In particular we note that

$$\begin{gathered} X \succ Y \Leftrightarrow X + R \succ Y + R, \qquad \forall R \in \mathbb{Z}[t, t^{-1}], \\ X \succ Y, \quad Z \succ 0 \Rightarrow X \cdot Z \succ Y \cdot Z \end{gathered}$$

**Definition 1.4.1.** Let $\mathbb{F}$ be a field. A graded $\mathbb{F}$-vector space, that is a vector space which decomposes into a direct sum $A_\bullet = \bigoplus_{n\in\mathbb{F}} A_n$, is said to be *admissible* if $\dim A_n < \infty$ and $A_n = 0$ for all $n \ll 0$. To such an $A_\bullet$ we define

$$P_{A_\bullet}(t) = \sum_n (\dim A_n)t^n \in \mathbb{Z}[t, t^{-1}] \tag{1.111}$$

which is called the *Poincaré series* associated to $A_\bullet$. ◁

**Lemma 1.4.1.** *Suppose we have a long exact sequence of admissible graded vector spaces $A_\bullet, B_\bullet, C_\bullet$:*

$$\longrightarrow A_k \xrightarrow{i_k} B_k \xrightarrow{j_k} C_k \xrightarrow{\partial_k} A_{k-1} \longrightarrow \cdots$$

*then $P_{A_\bullet} + P_{C_\bullet} \succ P_{B_\bullet}$.*

*Proof.* To simplify notation, we let

$$\begin{aligned} a_k &= \dim A_k, & b_k &= \dim B_k, & c_k &= \dim C_k \\ \alpha_k &= \dim\ker i_k, & \beta_k &= \dim\ker j_k, & \gamma_k &= \dim\ker \partial_k \end{aligned}$$

Then by [Ped00, Rank nullity theorem] we obtain $a_k = \dim A_k = \dim\ker i_k + \dim \operatorname{Im} i_k = \dim\ker i_k + \dim\ker j_k = \alpha_k + \beta_k$. Similarly we get $b_k = \beta_k + \gamma_k$ and $c_k = \gamma_k + \alpha_{k-1}$. Therefore

$$a_k - b_k + c_k = \alpha_k + \beta_k - (\beta_k + \gamma_k) + \gamma_k + \alpha_{k-1} = \alpha_k + \alpha_{k-1} \tag{1.112}$$

which gives us

$$P_{A_\bullet}(t) - P_{B_\bullet}(t) + P_{C_\bullet}(t) = \sum_k (a_k - b_k + c_k)t^k = \sum_k (\alpha_k + \alpha_{k-1})t^k = (1 + t)Q(t) \tag{1.113}$$

where $Q(t) = \sum_k a_k t^k$ and $a_k \geq 0$ for all $k$. Therefore $P_{A_\bullet}(t) + P_{C_\bullet}(t) = P_{B_\bullet}(t) + (1+t)Q(t)$, that is $P_{A_\bullet} + P_{C_\bullet} \succ P_{B_\bullet}$ as desired. □

### 1.4.2 Topological Morse Inequalities

For every compact topological space, $X$, we denote by $B_\lambda(X) = B_\lambda(X, \mathbb{F})$, the $\lambda$'th *Betti number* with coefficients in $\mathbb{F}$, that is $B_\lambda = \dim H_\lambda(X, \mathbb{F})$ and by $P_X(t) = P_{X,\mathbb{F}}(t)$ the *Poincaré polynomial*

$$P_{X,\mathbb{F}}(t) = \sum_\lambda B_\lambda(X, \mathbb{F}) t^\lambda \tag{1.114}$$

If $Y$ is a subspace of $X$, we define the relative Poincaré polynomial similarly. We can then write the *Euler characteristic* of $X$, see for example [Hat02, p. 146] for an introduction to this topological invariant, as follows

$$\chi(X) = \sum_\lambda (-1)^\lambda \dim H_\lambda(X, \mathbb{F}) = \sum_\lambda (-1)^\lambda B_\lambda(X) = P_X(-1) \tag{1.115}$$

**Proposition 1.4.1** (Topological Morse inequalities). *Let $f : M \to \mathbb{R}$ be a Morse function on a smooth compact manifold. Then*

$$P_f(t) \succ P_M(t) \tag{1.116}$$

*where $P_f$ is the Morse polynomial. In particular, $\chi(M) = \sum_\lambda (-1)^\lambda C_\lambda$.*

*Proof.* Now, let $f : M \to \mathbb{R}$ be a Morse function. Choose $a_0 < a_1 < \cdots < a_k$ such that $M^{a_i}$ contains precisely $i$ critical points and $M^{a_k} = M$. Note that this is possible by [Lemma 1.3.2]. From the long exact sequence of homology for the pair $(M^{a_i}, M^{a_{i-1}})$

$$\cdots \longrightarrow H_n(M^{a_{i-1}}) \longrightarrow H_n(M^{a_i}) \longrightarrow H_n(M^{a_i}, M^{a_{i-1}}) \longrightarrow H_{n-1}(M^{a_{i-1}}) \longrightarrow \cdots$$

we have $\bigoplus_n H_n(M^{a_i}) = H_\bullet(M^{a_i})$ and thus $P_{H_\bullet}(M^{a_i}) = \sum_n \dim H_n(M^{a_i}) t^n = \sum_\lambda B_\lambda(M^{a_i}) t^\lambda = P_{M^{a_i}}$. Likewise $P_{H_\bullet(M^{a_{i-1}})} = P_{M^{a_{i-1}}}$ and $P_{H_\bullet(M^{a_i}, M^{a_{i-1}})} = P_{M^{a_i}, M^{a_{i-1}}}$. Note that this makes sense because $M$ is compact. Indeed by [Corollary 1.1.2] there are only finitely many critical points and thus by [Main Theorem 1.3.3] we can give any sublevel set a finite CW-complex structure – in particular the various homology groups are finitely generated as required. Now by [Lemma 1.4.1] we get

$$P_{M^{a_{i-1}}} + P_{M^{a_i}, M^{a_{i-1}}} \succ P_{M^{a_i}} \tag{1.117}$$

Summing over all the sublevel sets we obtain

$$\sum_{i=1}^k P_{M^{a_{i-1}}} + \sum_{i=1}^k P_{M^{a_i}, M^{a_{i-1}}} \succ \sum_{i=1}^k P_{M^{a_i}} \qquad \text{thus by note 4} \qquad \sum_{i=1}^k P_{M^{a_i}, M^{a_{i-1}}} \succ P_M$$

where we have used that $M^{a_k} = M$. We claim that if $\lambda_i$ is the index if the critical point in $M^{a_i} - M^{a_{i-1}}$, then

$$\begin{aligned} H_n(M^{a_i}, M^{a_{i-1}}) &= H_n(M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}}) = H_n(e^{\lambda_i}, \partial e^{\lambda_i}) = H_n(D^{\lambda_i}, \partial D^{\lambda_i}) \\ &= \begin{cases} \mathbb{Z} & \text{if} \quad n = \lambda_i \\ 0 & \text{if} \quad n \neq \lambda_i \end{cases} \end{aligned} \tag{1.118}$$

Indeed, the first equality comes from [Strong Morse Principle] and the fact that homology is homotopy invariant, cf. [Hat02, Theorem 2.10]. Let $\epsilon(e^{\lambda_i})$ be an $\epsilon$-neighborhood of the cell. Noting that $\partial e^{\lambda_i} \simeq M^{a_{i-1}} \cap \epsilon(e^{\lambda_i})$ and the interiors cover $M^{a_{i-1}} \cup e^\lambda$, it follows by excision that the inclusion $(e^{\lambda_i}, \partial e^{\lambda_i}) \hookrightarrow (M^{a_{i-1}} \cup e^{\lambda_i}, M^{a_{i-1}})$ induces isomorphisms on homology for all $n$, cf. [Hat02, Excision theorem]. The last equation in (1.118) follows from the long exact sequence of reduced homology groups for the pair $(D^{\lambda_i}, \partial D^{\lambda_i})$. The maps $H_n(D^{\lambda_i}, \partial D^{\lambda_i}) \to \widetilde{H}_{n-1}(S^{\lambda_i - 1})$ are isomorphisms for all $n > 0$, since the remaining terms $\widetilde{H}_n(D_i^\lambda)$, where $D^{\lambda_i}$ is contractible, are zero for all $n$. Finally, $\widetilde{H}_{n-1}(S^{\lambda_i - 1}) = \mathbb{Z}$ exactly in the case $n = \lambda_i$, cf. [Hat02, Corollary 2.14], thus equation (1.118) holds. By definition of the relative Poincaré polynomial, we then have

$$P_f(t) = \sum_\lambda C_\lambda t^\lambda = \sum_{i=1}^k P_{M^{a_i}, M^{a_{i-1}}}(t) \succ P_M(t) \tag{1.119}$$

as desired. In particular, by definition of the order relation, we have $\chi(M) = P_M(-1) = P_f(-1) - (1-1)Q(-1) = P_f(-1) = \sum (-1)^\lambda C_\lambda$ as required. $\square$

**Corollary 1.4.1** (Weak Morse Inequalities). *Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold. Then $B_\lambda(M) \le C_\lambda$ for all $\lambda$.*

*Proof.* By the proposition we have $P_f(t) \succ P_M(t)$, that is $\sum_\lambda C_\lambda t^\lambda = \sum_\lambda B_\lambda(M)t^\lambda + (1+t)Q(t)$ where $Q \in \mathbb{Z}[t, t^{-1}]$ has non-negative coefficients. It follows that $C_\lambda \ge B_\lambda(M)$ for all $\lambda$. $\square$

**Corollary 1.4.2** (Morse's Lacunary Principle). *Let $f : M \to \mathbb{R}$ be a Morse function on a compact manifold $M$. If no consecutive powers of $t$ occur in the Morse polynomial $P_f(t)$, then $P_f(t) = P_M(t)$ for any coefficient field $\mathbb{F}$. In particular $M$ is torsion free.*

*Proof.* By the proposition we have $P_f(t) \succ P_M(t)$, that is $P_f(t) - P_M(t) = (1+t)Q(t)$. The first non-vanishing power of $t$ on the right hand side must occur in $P_f(t)$ as well. But if $Q(t)$ is not zero, the equation also implies that the next power occurs on the right and hence by the equation also on the left in $P_f(t)$, which contradicts the assumption. We conclude that $Q(t) \equiv 0$, that is $P_f(t) = P_M(t)$ as desired. $\square$

**Definition 1.4.2.** If $f : M \to \mathbb{R}$ is a Morse function such that $P_f(t) = P_M(t)$, then $f$ is called a *perfect* Morse function. $\lhd$

### The classical strong Morse inequalities

In the topological inequalities, when translating the order relation, we can think of $Q(t)$ as the $\mathbb{Z}$-error of $f$. The $(1+t)$ term on the right is what gives the inequality its power. Indeed, it is what feeds back information from the critical points of $f$ to the topology of $M$, as we saw with the Lacunary principle [Bot88]. As another example we note that the topological inequalities are actually equivalent to what Milnor calls the Morse inequalities, [Mil68].

**Theorem 1.4.1** (Morse Inequalities). *If $C_\lambda$ is the number of critical points of index $\lambda$ on the compact manifold $M$ and $B_\lambda(M)$ is the $\lambda$'th Betti number, then*

$$B_\lambda(M) - B_{\lambda-1}(M) + - \cdots \pm B_0(M) \le C_\lambda - C_{\lambda-1} + - \cdots \pm C_0 \tag{1.120}$$

*Proof.* From the topological Morse inequalities we know that $P_f(t) \succ P_M(t)$, thus there exists $Q \in \mathbb{Z}[t, t^{-1}]$ such that

$$P_f(t) = P_M(t) + (1+t)Q(t) \tag{1.121}$$

where $Q(t) = \sum_n q_n t^n$ and $q_n \ge 0$. Now, since we have

$$(1+t)\sum_{n\ge 0}(-1)^n t^n = \sum_{n\ge 0}(-1)^n t^n + \sum_{n\ge 1}(-1)^{n-1} t^n = 1 \tag{1.122}$$

we conclude that $(1+t)^{-1} = \sum_{n\ge 0}(-1)^n t^n$. We can then rewrite (1.121) as follows:

$$(1+t)^{-1}(P_f(t) - P_M(t)) = Q(t) \qquad \text{that is} \qquad \sum_{n\ge 0}(-1)^n t^n (P_f(t) - P_M(t)) = \sum_{n\ge 0} q_n t^n \tag{1.123}$$

Therefore we get

$$\sum_{n\ge 0} q_n t^n = \sum_\lambda \sum_{i\ge 0}(-1)^i C_\lambda t^{\lambda+i} - \sum_\lambda \sum_{i\ge 0}(-1)^i B_\lambda(M) t^{\lambda+i} \tag{1.124}$$

$$= \sum_{n\ge 0}(\sum_i (-1)^i C_{n-i} - \sum_i (-1)^i B_{n-i}(M))t^n \tag{1.125}$$

and thus $\sum_i (-1)^i C_{n-i} - \sum_i (-1)^i B_{n-i}(M) = q_n \ge 0$ or in other words

$$\sum_i (-1)^i C_{n-i} \ge \sum_i (-1)^i B_{n-i}(M) \qquad \text{for all } n \ge 0 \tag{1.126}$$

which is equivalent to the required expression (1.120). This completes the proof. $\square$

**Applications**

We pause for a moment to look at some basic, but intriguing, examples of how Morse theory works both ways: the homology groups of a manifold impose conditions on the critical points of any Morse function and, on the other hand, the critical point data of a Morse function sometimes allows us to compute the (co)homology groups.

If we let $f : S^2 \to \mathbb{R}$ be the height function, we have $P_f(t) = t^2 + 1$ and thus by [Morse's Lacunary Principle] we see that $P_{S^2}(t) = t^2 + 1$, hence we can then read off the homology groups since $B_\lambda(S^2) = \dim H_\lambda(S^2)$ which is 1 if $\lambda = 0, 2$ and 0 otherwise. In particular, $\chi(M) = P_M(-1) = 2$. The height function from [Figure 1.1.2] had the Morse polynomial $P_f(t) = 2t^2 + t + 1$. Since $M \simeq S^2$ we see that the corresponding Poincaré polynomial is $P_M(t) = t^2 + 1$ and thus $f$ is not a perfect Morse function on $M$.

From [Example 1.3.1] we have the CW-decomposition $e_0 \cup e^2 \cup \cdots \cup e^{2n}$ thus we immediately get the Morse polynomial $P_f(t) = t^{2n} + \cdots + t^2 + 1$. By [Morse's Lacunary Principle] we see that $f : \mathbb{C}P^n \to \mathbb{R}$ given by $f(z_0 : z_1 : \cdots : z_n) = \sum_{i=0}^n c_i |z_i|^n$ is a perfect Morse function, and this was why we could calculate the homology earlier.

Now, let us look at the canonical example in Morse theory, the torus, $T^2$, embedded vertically in $\mathbb{R}^3$. Since for the torus $T^2$ we have

$$H_\lambda(T^2) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & \lambda = 1 \\ \mathbb{Z} & \lambda = 0, 2 \\ 0 & \lambda \geq 3 \end{cases} \qquad \text{and thus} \qquad B_\lambda(T^2) = \begin{cases} 2 & \lambda = 1 \\ 1 & \lambda = 0, 2 \\ 0 & \lambda \geq 3 \end{cases} \tag{1.127}$$

cf. [Hat02, p. 106], we see by the weak Morse inequalities, that any Morse function on $T^2$ must have at least four critical points. Note moreover, the height function clearly also gives $P_f(t) = t^2 + 2t + 1$ hence is perfect.

For any embedding of $S^1$ in $\mathbb{R}^2$, the number of local maxima must equal the number of local minima. Indeed, we have $0 = \chi(S^1) = P_{S^1}(-1) = \sum_\lambda (-1)^\lambda C_\lambda = C_0 - C_1$. We have thus successfully expanded our knowledge of this simple example from the mere fact that there is at least one minima and one maxima. Relations like these holds through the Morse inequalities for more complicated examples, although the geometric perturbations naturally gets considerably more challenging to describe.

Suppose $M$ is a closed, orientable smooth manifold of odd dimension. Then any Morse function on $M$ has an even number of critical points. Indeed, the number of critical points is $P_f(1)$, and by [Proposition 1.4.1] there exists $Q \in \mathbb{Z}[t, t^{-1}]$ with non-negative coefficients such that

$$P_f(t) = P_M(t) + (1 + t)Q(t). \tag{1.128}$$

Since $M$ is odd dimensional and orientable we have $\chi(M) = 0$, and thus we get

$$P_f(-1) = P_M(-1) = \chi(M) = 0 \tag{1.129}$$

but we also have $P_f(1) \equiv P_f(-1) \mod 2$, thus $P_f(1) \in 2\mathbb{Z}$ as stated.

Based on these examples, one could imagine that every compact manifold posses a perfect Morse function. This is not the case however. This is a non-trivial fact, which, as it turns out, also depends on the coefficient field, [Gue02]. An example of a manifold which allows no perfect Morse functions is the Poincaré sphere, see [Nic07].

# 2 Morse-Bott Theory

Since Morse functions necessarily have isolated critical points, Morse theory immediately disqualifies many natural functions. The constant function is a trivial example, but there are innumerable other more interesting ones as well. In the present chapter, we will therefore engage in the quest of generalizing Morse theory to such cases. Furthermore, since finding a suitable Morse function is typically the hard part in applying Morse theory, weakening the requirements for such a function by generalizing the theory is a tremendous aid. This was first developed by Bott in the 1950's, see [Bot54], [Bot82].

## 2.1 Generalizing old acquaintances

In this section, we will introduce the natural generalizations of Morse functions, the set of critical points, the Morse lemma and the strong Morse principle from chapter 1, following [Nic07]. To see why one might want this, let's look at an illuminating example from [Gue02].

Consider the sphere $S^2 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$. Suppose we instead of the height function had $f : S^2 \to \mathbb{R}$ given by $f(x, y, z) = -z^2$. Then the critical points are $N = (0, 0, 1)$, $S = (0, 0, -1)$ and $E = \{(x, y, z) \,|\, z = 0\}$, that is the north pole, the south pole and the equator respectively. The first two have index 0, but to see what happens at the equator, we define local coordinates around $(1, 0, 0)$:

$$(\sqrt{1 - u^2 - v^2}, u, v) \mapsto (u, v) \tag{2.1}$$

Then we have $f(u, v) = -v^2$ and thus $f(0, 0) = 0$. From the Morse lemma, we see that the lack of a term $\pm u^2$ in this expression that we have degeneracy. The $u$-direction is the direction of the equator, and here $f$ is constant, so we are bound to have degeneracy in this direction. However, in the $v$-direction, the Morse lemma applies. From the integral curves of $-\nabla f$ one sees that we could hope for the following generalized cell decomposition:

$$S^2 \simeq (N \cup S) \cup_g (E \times [\tfrac{1}{2}, \tfrac{1}{2}]) \tag{2.2}$$

where $g : E \times \{-\frac{1}{2}, \frac{1}{2}\} \to N \cup S$ is an attaching map satisfying $g(E \times \{\frac{1}{2}\}) = N$ and $g(E \times -\frac{1}{2}) = S$. This is indeed the case, as we will see in the course of this section.

### 2.1.1 Non-degenerate critical manifolds

Since non-degenerate critical points for a Morse function $f$ are isolated by [Corollary 1.1.1], the set $Cr_f$ of critical points is a 0-dimensional manifold. What we aim for now is a generalization, where we consider a function $f : M \to \mathbb{R}$ whose critical set is a disjoint union $\bigsqcup_i N_i$ of connected submanifolds of dimension $d_i \geq 0$, called critical submanifolds.

We can, by definition cf. [BJ82, p. 33,38] write the restricted tangent bundle of $M$ to he submanifold $N_i$, that is $TM|_{N_i}$, as the following whitney sum:

$$TM|_{N_i} = TN_i \oplus \nu N_i \tag{2.3}$$

where $TN_i$ is the tangent bundle of $N_i$ and $\nu N_i$ is the normal bundle of $N_i$. If $N$ is $k$-dimensional a submanifold of $M$, then we can find a local coordinate system around $q \in N \subset M$ such that $x_{k+1} = 0, \ldots, x_m = 0$, cf. [BJ82, p. 9]. Then, for every $q \in N$ and every $v \in T_qN$ and $w \in T_qM$:

$$H_{f,q}(v, w) = 0 \tag{2.4}$$

since the extension $Y$ of $w$ satisfies $(Yf)(q) = Y_qf = D_{q,w}f = 0$, we see that the Hessian $H_{f,q}$ induces a bilinear symmetric form $\mathcal{H}_{f,q}$ on $\nu_qN \simeq T_qM/T_qN$.

**Definition 2.1.1.** Let $N$ be a smooth, compact and connected submanifold of $M$, such that every point of $N$ is a critical point of $f$. If for all $q \in N$, the induced Hessian $\mathcal{H}_{q,f}$ is nondegenerate, we call $N$ a *non-degenerate critical manifold* of $f$. $\triangleleft$

Equivalently, we could say that $N \subset M$ is a non-degenerate critical manifold if each point $p \in N$ is critical for $f$ and the Hessian $H_{f,p}$ is non-degenerate in the normal direction to $N$. Thus if $(x_1, \dots, x_m)$ is a local coordinate system around $p$ such that $N$ is given by the $m-k$ equations $x_{k+1} = 0, \dots, x_m = 0$, then

$$\det \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right) \Big|_p \neq 0 \qquad \text{for} \qquad i,j = k+1, \dots, m. \tag{2.5}$$

This structure furthermore gives us a way of decomposing the normal bundle $\nu N$ into a positive and a negative part as follows:

$$\nu N = \nu^+ N \oplus \nu^- N \tag{2.6}$$

where $\nu^+ N$ and $\nu^- N$ are spanned by the positive and negative eigenvectors of the Hessian of $f$ respectively. Note that these are topological invariants of $(N, f)$.

**Definition 2.1.2.** The function $f$ is called a *Morse-Bott function*, if its critical set consists of non-degenerate critical submanifolds. The dimension of $\nu^- N$ is called the Morse index of $N$ and is denoted by $\lambda(N)$. $\triangleleft$

Note that any Morse function is a Morse-Bott function, and a Morse-Bott functions is a Morse function exactly if each non-degenerate critical submanifold is a point.

### 2.1.2 The Morse Lemma with coefficients

With the terminology in place for this generalized setting, we can now prove the essential Morse lemma. Both its statement and proof looks remarkably familiar, though some complications must be taken care of. Indeed, in the article [BH04], it is the authors aim to fill in the details giving a complete proof of the Morse-Bott lemma, resting on a proof of the original Morse lemma by Palais, and Hirsch's lemma. We will therefore only provide a sketch proof here, and encourage the reader to consult this paper for more details.

**Theorem 2.1.1** (Morse-Bott Lemma). *Let $f : M \to \mathbb{R}$ be a Morse-Bott function, $N$ a connected component of $Cr(f)$ of dimension $k$, and $p \in N$. Then there exits an open neighborhood $U$ of $p$ and a smooth chart $\phi : U \to \mathbb{R}^k \times \mathbb{R}^{m-k}$, where $m = \dim M$, such that*

*(i)* $\phi(p) = 0$

*(ii)* $\phi(U \cap N) = \{(x, y) \in \mathbb{R}^k \times \mathbb{R}^{m-k} \,|\, y = 0\}$

*(iii)* $f \circ \phi^{-1}(x, y) = f(N) - y_1^2 - y_2^2 - \cdots - y_\lambda^2 + y_{\lambda+1}^2 + \cdots + y_{m-k}$

*where $\lambda \leq m-k$ is the index of $\mathcal{H}_{f,p}$ and $f(N)$ is the common value of $f$ on $N$.*

*Sketch proof.* The idea is to just generalize the proof of the original Morse lemma in a natural way. Let $N \subset M$ be a $k$-dimensional connected submanifold of $M$. By replacing $f$ with $f - c$ where $c$ is the common value of $f$ on $N$, we may assume $f(p) = 0$ for all $p \in N$. Then we can find a local coordinate system around $p \in N \subset M$ such that $x_{k+1} = 0, \dots, x_m = 0$, cf. [BJ82, p. 9]. This gives (i) and (ii), and we must prove that (iii) holds for this choice of $\phi$. By [Lemma 1.1.2] there exists smooth functions $g_1, \dots, g_m$ in a neighborhood of the origin such that

$$f(x_1, \dots, x_m) = \sum_{j=1}^{m} x_j g_j(x_1, \dots, x_m). \tag{2.7}$$

From the proof of [Lemma 1.1.2] it furthermore follows that

$$g_i(\mathbf{0}) = \int_0^1 \frac{\partial f}{\partial x_i}(\mathbf{0})\, dt = \frac{\partial f}{\partial x_i}(\mathbf{0}) = 0 \tag{2.8}$$

where the last equality follows since $p = \mathbf{0}$ is assumed to be a critical point, thus by [Lemma 1.1.2] again $g_j = \sum_{i=1}^{m} x_i h_{ij}$. Since we have $g_j(x_1, \ldots, x_m) = \int_0^1 \frac{\partial f}{\partial x_j}(tx_1, \ldots, tx_m)\, dt = 0$ and $h_{ij}(x_1, \ldots, x_m) = \int_0^1 \frac{\partial g_j}{\partial x_i}(tx_1, \ldots, tx_m)\, dt = 0$ for $j = 1, \ldots, k$ we obtain

$$f(x_1, \ldots, x_m) = \sum_{j=k+1}^{m} x_j g_j(x_1, \ldots, x_m) = \sum_{k+1 \leq i,j \leq m} x_i x_j h_{ij}(x_1, \ldots, x_m) = x^T A_x x \tag{2.9}$$

where $x$ is the transpose of $x^T = (x_k + 1, \ldots, x_m)$ and $A_x$ is the symmetric matrix with entries $a_{ij} = \frac{1}{2}(h_{ij}(x) + h_{ji}(x))$. By composing the chart $\phi$, chosen above, with a diffeomorphism, $\psi$, of $\mathbb{R}^k \times \mathbb{R}^{m-k}$ fixing the first component, we can assume that the Hessian in the direction normal to $\mathbb{R}^k \times \{0\}$ at $(0,0) \in \mathbb{R}^m$ for the expression $f \circ \phi^{-1}(x, y)$, that is

$$\Big(\frac{\partial^2 h}{\partial y_i \partial y_j}\Big|_{(0,0)}\Big), \tag{2.10}$$

is a diagonal matrix with the first $n$ diagonal entries equal to $-1$ and the rest equal to $+1$. Now, by assumption that $f$ is a Morse-Bott function, we know that for every $x \in \mathbb{R}^k$, the quadratic form

$$Q_x(y) = y^T\Big(\frac{\partial^2 h}{\partial y_i \partial y_j}\Big|_{(x,0)}\Big)y \tag{2.11}$$

is non-degenerate. The crucial part, resting on [BH04], is that given $x \in \mathbb{R}$ we can obtain a family of such diffeomorphisms $\psi_x$ depending *smoothly* on $x$, such that $\tilde{\phi}^{-1}(x, y) = \phi^{-1}(\psi_x(y)) : \mathbb{R}^k \times \mathbb{R}^{m-k} \to M$ is a chart where $f \circ \tilde{\phi}^{-1}(x, y) = \tilde{h}_x(y)$. This depends on Hirsch's lemma and Parlais proof of the Morse lemma. Now, the rest of the proof follows by induction on the number of terms in the generalized quadratic form of $f$, exactly as it was done in the proof of the Morse lemma. □

### 2.1.3 Generalized Morse principle

Recall that, given a vector bundle $V \to E \to B$, the *unit disc bundle* $D(E)$ of $E$ is the fibre bundle $\overline{B(0,1)} \to D(E) \to B$, where for each $b \in B$ we have $D(E)_b = p^{-1}(b) = \{v \in V \mid \|v\| \leq 1\}$, where the norm is with respect to some metric on $E$. We can now, by arguing exactly as in the proof of [Theorem 1.3.2], obtain the following generalized version of the strong Morse principle, which says that the manifold $M$ has the homotopy type of a "cell-bundle complex".

**Theorem 2.1.2** (Bott). *Let $f : M \to \mathbb{R}$ be an exhaustive Morse function. Suppose $f^{-1}(c)$ contains finitely many critical submanifolds $N_1, \ldots, N_k$. For $i = 1, \ldots, k$ let $D^-(N_i)$ be the closed unit disc bundle of the negative normal bundle $\nu^- N_i$. Then for $\epsilon > 0$ sufficiently small, we have the decomposition $M^{c+\epsilon} \simeq M^{c-\epsilon} \underset{\partial D^-(N_1)}{\cup} D^-(N_1) \ldots \underset{\partial D^-(N_k)}{\cup} D^-(N_k)$.*

It follows from [Theorem 2.1.2] that in particular, we have an isomorphism of graded homology

$$H_\bullet(M^{c+\epsilon}, M^{c-\epsilon}) = \bigoplus_{i=1}^{k} H_\bullet(D^-(N_i), \partial D^-(N_i)) \tag{2.12}$$

which we shall exploit to derive an analog of the Morse inequalities for certain, so called orientable, Morse-Bott functions. This will be the topic of the next section.

## 2.2 Orientable and completable Morse functions

In this section, we will introduce two properties for Morse-Bott functions; orientability and completability. The restriction to Morse-Bott functions with these properties ensures that the function is not too wild for us to be able to obtain the interaction between topology and analysis, we have seen in the previous chapter. In perticular, we will generalize the Morse inequalities to the setting of Morse-Bott theory and derive conditions sufficient to recognize perfect Morse-Bott functions.

### 2.2.1 Orientability

Recall that if $V$ is a vector space of dimension $k \geq 1$ we say that two ordered bases $(e_1, \ldots, e_k)$ and $(\tilde{e}_1, \ldots, \tilde{e}_k)$ are *consistently oriented* if the transition matrix $B$ defined by $e_i = B\tilde{e}_j$ has positive determinant. An *orientation* for $V$ is a equivalence class of ordered bases.

If $M$ is a smooth $m$-manifold, a *pointwise orientation* on $M$ is a choice of orientation for each tangent space $T_pM$ for $p \in M$. An *orientation* of $M$ is then a continuous pointwise orientation, and $M$ is *orientable*, if there exits an orientation for it, [Lee02, §13]. Alternatively, one could say that an orientation of $M$ is an orientation of the tangent bundle $TM$, cf. [BJ82, p. 37]. In particular we have the following lemma:

**Lemma 2.2.1.** *Every complex vector bundle is orientable as a real vector bundle.*

*Proof.* Suppose $V$ is a complex vector space of complex dimension $k$ with basis $\{e_i\}_{i=1}^k$. Then $V$ has a canonical orientation as a real vector space of dimension $2k$ with basis $\{e_1, ie_1, \ldots, e_k, ie_k\}$ over $\mathbb{R}$. The orientation determined by this basis is the canonical orientation for $V$. Now, if $\{\tilde{e}_i\}_{j=1}^k$ is another basis for $V$ over $\mathbb{C}$, $B$ is the complex change-of-basis matrix from $\{e_i\}$ to $\{\tilde{e}_j\}$ and $A$ is the real change-of-basis matrix from

$$\{e_1, ie_1, \ldots, e_k, ie_k\} \qquad \text{to} \qquad \{\tilde{e}_1, i\tilde{e}_1, \ldots, \tilde{e}_k, i\tilde{e}_k\} \tag{2.13}$$

then we have

$$\det A = \det B \cdot \overline{\det B} \geq 0. \tag{2.14}$$

We conclude that the two bases over $\mathbb{R}$ induced by the complex bases for $V$ determine the same orientation for $V$, and thus every complex vector bundle is orientable as a real vector bundle. $\square$

#### Thom isomorphism

There is an alternative definition of orientability, which in some cases is superior to the one given above, in the sense that despite its unpleasant appearance can be easier to apply for complicated spaces, see [Nic07].

**Definition 2.2.1.** Let $\mathbb{F}$ be a field. Given a real vector bundle $p : E \to B$ of rank $r$ over a compact CW-complex $B$, we say that $E$ is $\mathbb{F}$-orientable, if there exits a cohomology class

$$\tau \in H^r(D(E), \partial D(E); \mathbb{F}) \tag{2.15}$$

such that its restriction to each fiber, $(D(E)_b, \partial D(E)_b)$, for $b \in B$, defines a generator for the relative cohomology group $H^r(D(E)_b, \partial D(E)_b; \mathbb{F})$ . The class $\tau$ is called the *Thom class* of $E$ associated to the given orientation. $\lhd$

We need the classical result of the Thom Isomorphism Theorem, which plays a prominent role in more advanced Morse theory, as discovered by Bott in his proof of the Bott periodicity theorem.

**Theorem 2.2.1** (Thom Isomorphism Theorem). *If $\pi : E \to B$ of rank $r$ over $B$ is $\mathbb{F}$-orientable, then for any $k > 0$, the map*

$$H^k(B, \mathbb{F}) \to H^{k+r}(D(E), \partial D(E); \mathbb{F}) \qquad \textit{given by} \qquad \alpha \mapsto \tau_E \cup \pi^*\alpha \tag{2.16}$$

*is an isomorphism, called the* Thom isomorphism.

See also [Nic07]. By Poincaré duality, which one can actually also prove through Morse theory cf. [Mat02], we see that we have an equivalent isomorphism

$$H_{k+r}(D(E), \partial D(E); \mathbb{F}) \simeq H_k(X; \mathbb{F}). \tag{2.17}$$

For a proof of the Thom Isomoprhism Theorem, we refer the reader to [MT97, § 21], which also explains orientations classes for oriented vector bundles thoroughly.

**Definition 2.2.2.** Let $\mathbb{F}$ be a field. A *Morse-Bott function* $f : M \to \mathbb{R}$ is called $\mathbb{F}$-orientable if for every critical submanifold $N$ the negative normal bundle $\nu^- N$ is $\mathbb{F}$-orientable. $\lhd$

### 2.2.2 Morse-Bott inequalities

We will now generalize the Morse inequalities to the more general setting of Morse-Bott functions, again inspired by [Nic07]. Unfortunately, we need the Thom isomorphism theorem, thus only the special class of orientable Morse-Bott functions qualify. First of all, we need to make sense of the generalized Morse polynomial.

**Definition 2.2.3.** Let $\mathbb{F}$ be a field. The $\mathbb{F}$-*Morse-Bott-polynomial* of a Morse-Bott function on a compact manifold $M$ is the polynomial

$$P_f(t) = P_f(t;\mathbb{F}) = \sum_N t^{\lambda(N)} P_{N,\mathbb{F}}(t) \tag{2.18}$$

where the summation is over all critical submanifolds of $M$. ◁

Note that this coincides with our previous definition of the Morse polynomial. Indeed, if $f$ is a Morse function, and we have $k$ critical points, then the non-degenerate submanifolds are points and hence $B_\lambda(N) = \dim H_\lambda(N) = 1$ if $\lambda = 0$ and 0 otherwise. This gives $P_f(t) = \sum_{p\in Cr(f)} t^{\lambda(p)} 1 t^0 = \sum_{i=1}^k t^{\lambda_i}$ which was [Definition 4] as stated.

**Corollary 2.2.1** (Morse-Bott inequalities)**.** *Suppose $f : M \to \mathbb{R}$ is an $\mathbb{F}$-orientable Morse-Bott function on a compact manifold $M$. Then we have the Morse-Bott inequalities*

$$P_f(t) \succ P_{M,\mathbb{F}}(t) \tag{2.19}$$

*in particular $\chi(M) = \sum_S (-1)^{\lambda(f,S)} \chi(S)$.*

*Proof.* Let $f : M \to \mathbb{R}$ be an $\mathbb{F}$-orientable Morse-Bott function. Choose $a_0 < a_1 < \cdots < a_k$ such that $M^{a_i}$ contains precisely $i$ non-degenerate critical submanifolds and $M^{a_k} = M$. Note that this is possible by [Lemma 1.3.2]. Now, since $\dim \nu^- N_i = \lambda(N_i)$, we get by (2.17), suppressing the field $\mathbb{F}$ for simplicity:

$$P_{D^-(N_i),\partial D^-(N_i)}(t) = \sum_{k\geq 0} \dim H_k(D^-(N_i), \partial D^-(N_i)) t^k = \sum_{k\geq 0} \dim H_{k-\lambda(N_i)}(N_i) t^k \tag{2.20}$$

$$= \sum_{k\geq 0} \dim H_k(N_i) t^{k+\lambda(N_i)} = t^{\lambda(N_i)} P_{N_i}(t) \tag{2.21}$$

Now, summing over all non-degenerate critical submanifolds of $M$, we have

$$P_f(t) = \sum_N t^{\lambda(N)} P_N(t) = \sum_N P_{D^-(N),\partial D^-(N)}(t) = \sum_{i=1}^k P_{M_i,M_{k-1}}(t) \succ P_M(t) \tag{2.22}$$

where we have used (2.12) and the result $\sum_{i=1}^k P_{M^{a_i},M^{a_{i-1}}} \succ P_M$ from the proof of the standard Morse inequalities. This is the desired result. □

### 2.2.3 Completable Morse functions

As a final restriction on Morse-Bott functions, we will introduce the notion of completability. In essence, this ensures that the way in which we attach the "cell-bundles" together is nice enough for us to obtain perfect Morse-Bott functions, and thus extract the homology of the underlying manifold.

**Definition 2.2.4.** Let $f : M \to \mathbb{R}$ be a Morse-Bott function on a compact manifold $M$. If for every critical value $c \in \mathbb{R}$ and every critical submanifold $N \subset f^{-1}(c)$ the inclusion $\partial D^-(N) \to M^{c-\epsilon}$ induces the trivial morphism in homology, then $f$ is called *completable.* ◁

**Proposition 2.2.1.** *Let $f : M \to \mathbb{R}$ be a completable, $\mathbb{F}$-orientable Morse-Bott function on a compact manifold. Then $f$ is perfect.*

*Proof.* Suppose $f : M \to \mathbb{R}$ is a Morse-Bott function on the compact, smooth manifold $M$. Denote by $c_1 < \cdots < c_\nu$the critical values of $M$ and set

$$a_0 = c_1 - 1, \qquad a_\nu = c_\nu + 1 \qquad \text{and} \qquad a_i = \frac{c_i + c_{i+1}}{2}, i = 1, \ldots, \nu - 1 \tag{2.23}$$

We denote by $\mathcal{N}_i \subset Cr(f)$ the set of critical manifolds on the level set $\{p \in M \mid f(p) = c_i\}$. From (2.12) and the arguments from the proof of the Topological Morse Inequalities, we have

$$H_\bullet(M^i, M^{i-1}) = \bigoplus_{N \in \mathcal{N}_i} H_\bullet(D^-(N), \partial D^-(N)) \tag{2.24}$$

Also, as we have seen before, $H_\lambda(D^-(N), \partial D^-(N)) = \mathbb{Z}$ while $H_\lambda(D^-(N), \partial D^-(N)) = 0$ for $k \neq \lambda$. The connecting morphism $H_\bullet(M^i, M^{i-1}) \xrightarrow{\partial} H_{\bullet-1}(M^{i-1})$ is trivial by assumption of $f$ being completable hence for every $1 \leq i \leq \nu$ we obtain the following short exact sequence:

$$0 \longrightarrow H_\bullet(M^{i-1}) \longrightarrow H_\bullet(M^i) \longrightarrow \bigoplus_{N \in \mathcal{N}_i} H_\bullet(D^-(N), \partial D^-(N)) \longrightarrow 0$$

We conclude that we have

$$P_{M^i}(t) = P_{M^{i-1}}(t) + \sum_N t^{\lambda(N)} P_N(t) \tag{2.25}$$

Summing over $i = 1, \ldots, \nu$ we obtain, since $M^0 = \emptyset$ and $M^\nu = M$:

$$P_M(t) = \sum_{i=1}^{\nu} \sum_{N \in \mathcal{N}_i} t^{\lambda(N)} P_N(t) = P_f(t) \tag{2.26}$$

which was the desired result. □

**Corollary 2.2.2.** *Suppose $f$ is an orientable Morse-Bott function for which every critical submanifold $N$ we have $\lambda \in 2\mathbb{Z}$ and $P_N(t)$ is even, then $f$ is perfect, that is $P_f(t) = P_M(t)$.*

*Proof.* We will prove the statement by induction on $k$. By [Proposition 2.2.1] it suffices to prove that $f$ is completable, or in other words that the inclusion $\partial D^-(N) \to M^{c-\epsilon}$ induces the trivial morphism in homology for every critical value $c \in \mathbb{R}$. By [Theorem 2.1.2] we have the decomposition

$$M_k \simeq D^-(N_1) \cup \cdots \cup D^-(N_k) \tag{2.27}$$

By assumption $B_\lambda(N_1) = B_\lambda(M^1) = 0$ if $\lambda$ is odd. Now suppose this is true for $B_\lambda(M_{k-1})$. We have the long exact sequence in homology:

$$\cdots \longrightarrow H_{\lambda+1}(M_k, M_{k-1}) \longrightarrow H_\lambda(M_{k-1}) \longrightarrow H_\lambda(M_k) \longrightarrow H_\lambda(M_k, M_{k-1}) \longrightarrow H_{\lambda-1}(M_{k-1}) \longrightarrow \cdots$$

from which we deduce that $B_\lambda(M_k) = 0$ if $\lambda$ is odd. Therefore, if $\lambda$ is even we get the short exact sequence:

$$0 \longrightarrow H_\lambda(M_{k-1}) \longrightarrow H_\lambda(M_k) \longrightarrow H_\lambda(M_k, M_{k-1}) \longrightarrow 0$$

We conclude that $f$ is indeed a perfect Morse-Bott function as desired. □

# 3 Cohomology of Grassmannians

In this section we apply the theory developed in the last couple of sections to investigate the topology of complex Grassmannians, $Gr_k(\mathbb{C})$. These spaces are important for a number of reasons. Firstly, by giving a collection of subspaces of some vector space a smooth structure, we can talk about a smooth choice of subspaces or open and closed collections of subspaces. Thereby, one can describe ideas that could not be considered otherwise – or at least describe them more efficiently.

A natural example comes from tangent bundles of smooth manifolds embedded in Euclidean space. Suppose we have a manifold $M$ of dimension $k$ embedded in $\mathbb{C}^n$. At each point $p \in M$, the tangent space $T_pM$ can be considered as a subspace of $\mathbb{C}^n$. The map assigning to $p$ its tangent space thus defines a map $M \to Gr_k(\mathbb{C}^n)$.

Secondly, Grassmannians pop up in many diverse areas of mathematics such as algebraic geometry, differential geometry and they provide classifying spaces in $K$-theory. Furthermore, they have found applications in computer vision, and equilibrium theory in economics.

## 3.1 Analysis of Grassmannian

**Definition 3.1.1.** Let $V$ be a finite dimensional vector space. The *Grassmann manifold*, or *Grassmannian*, $Gr_k(V)$, is the space of $k$-dimensional linear subspaces of $V$. The complex Grassmannian is the space $Gr_k(\mathbb{C}^n)$, which we will denote $G_{k,n}$ for short.

We note, that $Gr_1(\mathbb{R}^n)$ is just the familiar real projective space $\mathbb{R}P^n$ and likewise $Gr_1(\mathbb{C}^n) = G_{1,n}$ is the complex projective space $\mathbb{C}P^n$ which we studied earlier. With the Grassmannian, we are now allowed to look at subspaces of any dimension, and over any arbitrary vector space, thus clearly we have a huge class of spaces to describe. In order for Morse theory to be of any help in this matter, we need to know that Grassmannians are actually manifolds. This is, luckily, true, and there are quite a number of ways to prove this fact, cf. [Lee02]. We will take one of the more unusual approaches, with the purpose of making it possible for us to define a suitable Morse function on these spaces later on.

### 3.1.1 The Grassmannian as a manifold

We need to introduce the terminology of Hermitian vector spaces and some related notation, to set the stage for the proof of the Grassmannians being manifolds. This section is inspired from [Nic07].

**Definition 3.1.2** (Hermitian)**.** A *Hermitian metric* on a complex vector space $W$ is a positive-definite Hermitian form on $W$, that is a function $m : W \times W \to \mathbb{C}$ such that for all $u, v, w \in W$ and $a, b \in \mathbb{R}$

$$h(au + bv, w) = ah(u, w) + bh(v, w) \qquad \text{and} \qquad h(u, v) = \overline{h(v, u)} \tag{3.1}$$

such that for all $w \in W\backslash\{0\}$ we have $h(w) > 0$. This is also called a positive definite, symmetric sesquilinear form. We denote $h(\cdot, \cdot)$ by $\langle \cdot, \cdot \rangle$. A complex vector space $W$ equipped with a Hermitian metric, $(W, h)$, is called a *Hermitian vector space*. For every Hermitian vector space $W$, let $L(W)$ denote the linear space of Hermitian linear operators $T : W \to W$, that is $T$ is linear and satisfies $\langle Tx, y \rangle = \langle x, Ty \rangle$. $\lhd$

**Proposition 3.1.1.** *The complex Grassmannian $G_{k,n}$ is a complex, compact manifold of complex dimension $k(n-k)$. Furtheromore, we have a diffeomorphism $G_{k,n} \simeq G_{n-k,n}$.*

*Proof.* First of all, we need to topologize $Gr_k(\mathbb{C}^n)$. To do this, denote for every $U \in Gr_k(\mathbb{C}^n)$ by $P_U : \mathbb{C}^n \to \mathbb{C}^n$, the orthogonal projection onto $U$. We give $Gr_k(\mathbb{C}^n)$ the metric topology induced by the metric

$$d(U_1, U_2) = \|P_{U_1} - P_{U_2}\| \tag{3.2}$$

where $\|\cdot\|$ denotes the operator norm on $L(\mathbb{C}^n)$, see [Mun00, § 20], [Fol99]. Note that for any $U_1, U_2 \in Gr_k(\mathbb{C}^n)$ we have $d(U_1, U_2) = \sup\{|P_{U_1}z - P_{U_2}z| \,:\, |z| = 1\} \leq 1$, thus we can view $Gr_k(\mathbb{C})$ as a bounded and closed subset of $\mathbb{C}^{n^2}$, i.e. it is compact. Since every compact metrizable space is Hausdorff, we conclude that in particular, $Gr_k(\mathbb{C}^n)$ is compact Hausdorff.

Next, we need to find local coordinates. Suppose, $U \in Gr_k(\mathbb{C}^n)$ and $S : U \to U^\perp$ is a linear map. Denote by $\Gamma_S \in G_k(\mathbb{C}^n)$ the graph of the operator $S$, that is

$$\Gamma_S = \{(x, y) \in U \times U^\perp \,|\, y = Sx\} = \{x + Sx \,|\, x \in U\} \subset U \oplus U^\perp = \mathbb{C}^n \tag{3.3}$$

cf. [Fol99, p. 162]. Then we have the following smooth map

$$\eta : Hom(U, U^\perp) \hookrightarrow Gr_k(\mathbb{C}^n) \qquad \text{given by} \qquad \eta(S) = \Gamma_S \tag{3.4}$$

mapping onto the open subset $\mathcal{U} \subset Gr_k(\mathbb{C}^n)$ consisting of all $k$-planes intersecting $U^\perp$ transversally. Note that since $U \simeq \mathbb{C}^k$ we have $Hom(U, U^\perp) \simeq M_{k,n-k}(\mathbb{C}) \simeq \mathbb{C}^{k(n-k)}$. Likewise, we claim that we have a smooth map

$$\kappa : Gr_k(\mathbb{C}^n) \hookrightarrow M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2} \qquad \text{given by} \qquad \kappa(U) = P_U \tag{3.5}$$

and that then $\eta^{-1} : \mathcal{U} \to \mathbb{C}^{k(n-k)}$ defines local coordinates, which we will call *graph coordinates*, on $Gr_k(\mathbb{C}^n)$ near $U \in \mathcal{U}$. Summarizing we have:

$$\mathbb{C}^{k(n-k)} \simeq Hom(U, U^\perp) \xrightarrow{\ \eta\ } Gr_k(\mathbb{C}^n) \xrightarrow{\ \kappa\ } M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$$

To verify the claim, it suffices to show that $\kappa \circ \eta$ is an embedding. Note first that if we let $S^* : U^\perp \to U$ denote the adjoint operator, which is well defined due to $\mathbb{C}^n$ being a Hilbert space, then for every $S \in Hom(U, U^\perp)$ we have

$$\Gamma_S^\perp = \{-y + S^*y \,|\, y \in U^\perp\} \subset U^\perp \oplus U \tag{3.6}$$

Indeed, using the fact that $S^*y \in U$ and $Sx \in U^\perp$, we have $\langle x + Sx, -y + S^*y\rangle = -\langle x, y\rangle - \langle Sx, y\rangle + \langle x, S^*y\rangle + \langle Sx, S^*y\rangle = -\langle Sx, y\rangle + \langle Sx, y\rangle = 0$. Then we can write $\eta \circ \kappa(S) = P_{\Gamma_S}$ in terms of $P_U$ and $S$. Let $v = P_U v + P_{U^\perp} v \in \mathbb{C}^n$. Then we have $P_{\Gamma_S} v = x + Sx$ for $x \in U$ if and only if $v - (x - Sx) \in \Gamma_S^\perp$, that is if and only if there exits $x \in U$ and $y \in U^\perp$ such that

$$\begin{cases} x + S^* = P_U v \\ Sx - y = P_{U^\perp} v \end{cases} \tag{3.7}$$

If we define the operator $\mathcal{S} : U \oplus U^\perp \to U \oplus U^\perp$ with the block decomposition:

$$\mathcal{S} = \begin{bmatrix} 1_U & S^* \\ S & -1_{U^\perp} \end{bmatrix} \qquad \text{then (3.7) can be written} \qquad \mathcal{S}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} P_U v \\ P_{U^\perp} v \end{bmatrix} \tag{3.8}$$

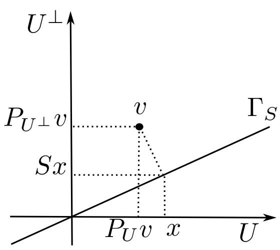



Figure 3.1: Subspaces as graphs of linear operators

Note that we have

$$\mathcal{S}^2 = \begin{bmatrix} 1_U + S^*S & 1_U S^* - S^* 1_{U^\perp} \\ S 1_U - 1_{U^\perp} S & SS^* + 1_{U^\perp} \end{bmatrix} = \begin{bmatrix} 1_U + S^*S & 0 \\ 0 & SS^* + 1_{U^\perp} \end{bmatrix} \tag{3.9}$$

We conclude that $\mathcal{S}$ is invertible and has the expression:

$$\mathcal{S}^{-1} = (\mathcal{S}^2)^{-1}\mathcal{S} = \begin{bmatrix} (1_U + S^*S)^{-1} & 0 \\ 0 & (SS^* + 1_{U^\perp})^{-1} \end{bmatrix} \cdot \begin{bmatrix} 1_U & S^* \\ S & -1_{U^\perp} \end{bmatrix} \tag{3.10}$$

$$= \begin{bmatrix} (1_U + S^*S)^{-1} & (1_U + S^*S)^{-1}S^* \\ (SS^* + 1_{U^\perp})^{-1}S & -(SS^* + 1_{U^\perp})^{-1} \end{bmatrix} \tag{3.11}$$

We can thus solve the system (3.8), and obtain the following expression for $x$:

$$x = (1_U + S^*S)^{-1}P_U v + (1_U + S^*S)^{-1}S^*P_{U^\perp}v. \tag{3.12}$$

Thus we have $P_{\Gamma_S}v = (x, Sx) = (1_U, S)x$, and therefore we see that we finally get the desired result:

$$P_{\Gamma_S} = \begin{bmatrix} 1_U \\ S \end{bmatrix} \begin{bmatrix} (1_U + S^*S)^{-1} & (1_U + S^*S)^{-1}S^* \end{bmatrix} \tag{3.13}$$

$$= \begin{bmatrix} (1_U + S^*S)^{-1} & (1_U + S^*S)^{-1}S^* \\ S(1_U + S^*S)^{-1} & S(1_U + S^*S)^{-1}S^* \end{bmatrix} \tag{3.14}$$

which is indeed a smooth and injective map. Finally, since $rank_S(\eta\circ\kappa) = k(n-k) = \dim Hom(U, U^\perp)$, we can write

$$\eta\circ\kappa : \mathbb{C}^{k(n-k)} \to \eta\circ\kappa(\mathbb{C}^{k(n-k)}) \subset \mathbb{C}^{n^2}, \qquad (z_1, \ldots, z_{k(n-k)}) \mapsto (z_1, \ldots, z_{k(n-k)}, 0, \ldots, 0). \tag{3.15}$$

Since its differential is then an isomorphism, we conclude by [BJ82, Inverse function theorem] that we have a diffeomorphism. Finally, we conclude by [BJ82, Theorem 5.7] that $\eta\circ\kappa$ is indeed an embedding as desired.

The last claim, that we have a diffeomorphism $G_{k,n} \to G_{n-k,n}$, follows easily by letting it be the map which associates to each $k$-dimensional subspace its orthogonal complement with respect to a fixed Hermitian metric on the ambient space. Indeed, it is its own inverse and smooth. This completes the proof. □

### 3.1.2 Morse functions on Grassmannians

Now that we have a smooth, compact manifold at our disposal, we are in a position to apply the derived Morse theory from the last couple of chapters. We aim to find a Morse-Bott function on $Gr_k(\mathbb{C})$, allowing us to prove the following proposition:

**Proposition 3.1.2.** *For every $1 \le k \le n$ the Poincaré polynomial $P_{k,n}(t)$ is even, that is the odd Betti numbers of $G_{k,n}$ are trivial. Moreover*

$$P_{k,n+1}(t) = P_{k,n}(t) + t^{2(n+1-k)}P_{k-1,n}(t) \tag{3.16}$$

We will now engage in the, rather lengthy, quest of finding a suitable Morse function, exploiting the construction from the last section, following [Nic07]. Luckily, however, once we have found a suitable Morse function, the derived theory ends the proof.

**Lemma 3.1.1.** *Let $W$ be a complex Hermitian vector space. The map*

$$\Omega : Gr_k(W) \to L(W), \qquad \textit{defined by} \qquad \Omega(U) = P_U \tag{3.17}$$

*where $P_U$ is the orthogonal projection on $U$, is a smooth embedding.*

*Proof.* The map $\kappa$ from the previous lemma, applied to the space $W$, will do the job. □

We can obtain real-valued functions on $Gr_k(\mathbb{C}^n)$ by taking "height functions" with respect to this embedding. The next lemma shows that there is a particularly nice one such function.

**Lemma 3.1.2.** *Let $V$ be a complex $n$-dimensional vector space equipped with a Hermitian metric. The function $f : Gr_k(W) \to \mathbb{R}$ defined by*

$$f(U) = \operatorname{Re} tr(P_{\mathbb{C}}P_U) \tag{3.18}$$

*where $P_{\mathbb{C}} : \mathbb{C}\oplus V \to \mathbb{C}\oplus V$ is the orthogonal projection onto $\mathbb{C}$, is an orientable Morse-Bott function.*

*Proof.* Let us define $W = \mathbb{C} \oplus V$. Then $W$ is a Hermitian vector space, since we can obtain a metric by taking the direct sum of the metrics on $V$ and $\mathbb{C}$. Then the map $g : L(\mathbb{C} \oplus V) \to \mathbb{R}$ defined by $g(T) = \mathrm{Re}\, tr(P_{\mathbb{C}} T)$ is a well defined, smooth function. Composing $g$ with the smooth embedding $U \mapsto P_U$ from [Lemma 3.1.1] we obtain $f$, which is then a well defined smooth function. Note that we can write[1]

$$f(U) = \langle P_U e_0, e_0 \rangle. \tag{3.19}$$

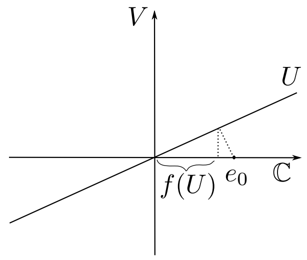



Figure 3.2: The map $f : Gr_k(W) \to \mathbb{R}$ given by $f(U) = \langle P_U e_0, e_0 \rangle$

Indeed, note that if we let $u_1, \ldots u_k$ be an orthonormal basis for $U$ and denote the $i$'th coordinate of vector $u_\alpha$ by $u_\alpha(i)$, then we have

$$P_U = \begin{bmatrix} u_1 & \ldots & u_k \end{bmatrix} \begin{bmatrix} u_1 & \ldots & u_k \end{bmatrix}^T = \Big[ \sum_{\alpha=1}^{k} u_\alpha(i) u_\alpha(j) \Big]_{1 \le i,j \le n} \tag{3.20}$$

$$P_{\mathbb{C}} = e_0 e_0^T = [\beta_{i,j}]_{1 \le i,j \le n} \qquad \text{where } \beta_{i,j} = 1 \text{ for } i,j = n \text{ and } 0 \text{ otherwise} \tag{3.21}$$

$$P_{\mathbb{C}} P_U = \begin{pmatrix} 0 & \ldots & 0 \\ \vdots & & \vdots \\ \sum_{\alpha=1}^{n} u_\alpha(n) u_\alpha(1) & \ldots & \sum_{\alpha=1}^{n} u_\alpha(n) u_\alpha(n) \end{pmatrix} \tag{3.22}$$

This finally gives us the stated identity:

$$\mathrm{Re}\, tr(P_{\mathbb{C}} P_U) = \mathrm{Re} \sum_{\alpha=1}^{n} u_\alpha(n) u_\alpha(n) = \langle P_U e_0, e_0 \rangle \tag{3.23}$$

By [Definition 2.1.1,2.1.2] we must prove that its set of critical points consists of a disjoint union of non-degenerate critical manifolds. We start by proving the following three properties:

(i) $f$ satisfies $0 \le f \le 1$, $\forall U \in Gr_k(W)$ and $f^{-1}(0) = Gr_k(V)$, $f^{-1}(1) = Gr_{k-1}(V)$

*Proof of (i).* Let $U \in W$ be a $k$-dimensional linear subspace. Then since $\|P_U\| = 1$, being an orthogonal projection, we have $\|P_U e_0\| \le 1$ in particular $0 \le \langle P_U e_0, e_0 \rangle \le \langle e_0, e_0 \rangle = 1$.

Since $f(U) = \langle P_U e_0, e_0 \rangle = 0$ if and only if $P_U e_0 = 0$, that is if and only if $e_0 \in U^\perp$ we conclude that $U \subset e_0^\perp = V$. Therefore $L \in Gr_k(V)$. For the other pre-image, note that we have a natural embedding $Gr_{k-1}(V) \to Gr_k(W)$ defined by $U \mapsto \mathbb{C}e_0 \oplus U$. Now, $f(U) = \langle P_U e_0, e_0 \rangle = 1$ if and only if $P_U e_0 = e_0$, that is $e_0 \in U$ thus $U \in Gr_{k-1}(V)$ by the embedding. □

(ii) The only critical values of $f$ are 0 and 1.

*Proof of (ii).* Since 0 and 1 are extremal values of $f$, these are critical values. To see that these are the only ones, let $U \in Gr_k(W)$ be given such that $0 < f(U) < 1$. We wish to show that then $U$ is a regular point of $f$. From the condition $0 < f(U) < 1$ it follows, that $U$ intersects the hyperplane $V \subset W$ transversally along a $k-1$-dimensional linear subspace $U' \subset U$.

[1] This is also equivalent to $\cos \angle(e_0, U)$, see [Nic07].

Pick an orthonormal basis $e_1, \dots, e_{k-1}$ for $U'$. We can extend this to an orthonormal basis $e_1, \dots, e_{k-1}, \dots, e_n$ for all of $V$. Then we have

$$U = U' + \mathbb{C}v \qquad \text{where} \qquad v = c_0 e_0 + \sum_{j \geq k} c_j e_j, \quad \text{and} \quad |c_0|^2 + \sum_{j \geq k} |c_j|^2 = 1 \tag{3.24}$$

and thus $\langle P_U e_0, e_0 \rangle = \langle |u_0|^2 e_0 + \sum_{j \geq k} |c_j|^2 e_0, e_0 \rangle = |c_0|^2$ by orthogonality. Now, choosing

$$v(t) = a_0(t) e_0 + \sum_{j \geq k} a_j(t) e_j, \quad \text{where} \quad |a_0(t)|^2 + \sum_{j \geq k} |a_j(t)|^2 = 1 \tag{3.25}$$

where $a_0, a_j : \mathbb{R} \to \mathbb{C}$ are smooth functions satisfying $\frac{d}{dt}|a_0(t)|^2|_{t=0} \neq 0$ and $a_0(0) = c_0$ we obtain a smooth path

$$\gamma : I \to G_k(W) \qquad \text{given by} \qquad \gamma(t) = U_t = U' + \mathbb{C}v(t) \tag{3.26}$$

where $\frac{d}{dt} f(U_t)|_{t=0} = \frac{d}{dt}(P_{U_t} e_0, e_0)|_{t=0} = \frac{d}{dt}|a_0(t)|^2|_{t=0} \neq 0$. Therefore $U_0 = U$ is a regular point of $f$ as desired. We conclude that 0 and 1 are indeed the only critical values. □

(iii) The level sets $S_i = f^{-1}(i)$, $i = 0, 1$ are non-degenerate critical manifolds.

*Proof of (iii).* Since $S_0 = Gr_k(V)$ by (i), it is a complex submanifold of $Gr_k(W)$ of complex dimension $k(n-k)$, cf. [Proposition 3.1.1]. Similarly $S_1 = Gr_{k-1}(V)$ is a submanifold of complex dimension $(k-1)(n-k+1)$. Therefore, their complex codimensions are

$$\begin{aligned} codim_{\mathbb{C}}(S_0) &= \dim Gr_k(W) - \dim Gr_k(V) = k(n+1-k) - k(n-k) = k \\ codim_{\mathbb{C}}(S_1) &= \dim Gr_k(W) - \dim Gr_k(V) = k(n+1-k) - (k-1)(n-k+1) = n-k+1 \end{aligned}$$

To prove that $S_0$ is a non-degenerate critical manifold, it suffices to prove that for every $U \in S_0 = Gr_k(V) \subset W$ there exists a smooth map $\Phi : \mathbb{C}^k \to Gr_k(W)$ which is an immersion at 0 such that furthermore

$$\Phi(0) = U, \qquad \text{and} \qquad f \circ \Phi(0) \quad \text{is a non-degenerate minimum} \tag{3.27}$$

For $v \in V$ let $X_v : W \to W$ be the skew-Hermitian linear operator defined by

$$X_v(e_0) = v, \qquad X_v(w) = -\langle w, v \rangle e_o \qquad \forall w \in W \tag{3.28}$$

Then for $a, b \in \mathbb{R}$ we have $X_v(aw_1 + bw_2) = -\langle aw_1 + bw_2, v \rangle e_0 = -a\langle w_1, v \rangle e_0 - b\langle w_2, v \rangle e_0 = aX_v(w_1) + bX_v(w_2)$, and $X_{tv} = -\langle w, tv \rangle e_0 = -t\langle w, v \rangle e_0 = tX_v$, thus the map $v \mapsto X_v \in L(W)$ is $\mathbb{R}$-linear. Furthermore, we can form the related linear operator $e^{tX_v} : W \to W$. Now we set

$$\Phi(v) = e^{X_v} U \qquad \text{and} \qquad P(v) = P_{\Phi(v)} \tag{3.29}$$

and observe that then we have[2]

$$\begin{aligned} P(v) &= e^{X_v} P_U e^{-X_v} \qquad \text{and} & (3.30) \\ \dot{P}_v &= \frac{dP(tv)}{dt}|_{t=0} = \frac{d}{dt}(e^{tX_v} P_U e^{-tX_v})|_{t=0} & (3.31) \\ &= (X_v e^{tX_v} P_U e^{-tX_v} + e^{tX_v} P_U (-X_v) e^{-tX_v})|_{t=0} = X_v P_U - P_U X_v = [X_v, P_U] & (3.32) \end{aligned}$$

This, in turn, gives us the following, if $v \in U$:

$$\langle \dot{P}_v e_0, v \rangle = \langle X_v P_U e_0 - P_U X_v e_0, v \rangle = -\langle P_U X_v e_0, v \rangle = -\langle v, v \rangle = -|v|^2 \tag{3.33}$$

since we have $e_0 \in U^\perp$ and thus $P_U e_0 = 0$. We conclude that if $\dot{P}_v = 0$ then so is $v$. This proves that the map

$$\Phi : \mathbb{C}^k \supset U \to Gr_k(W) \tag{3.34}$$

[2]The first identity follows by straightforward computations using the definitions and that $e^{X_v} = \sum_{n=0}^{\infty} \frac{X_v^n}{n!}$.

is an immersion at $v = 0$. To see that $f \circ \Phi(0)$ is a non-degenerate minimum, we first compute this composition:

$$f(\Phi(v)) = \langle P_{\Phi(v)} e_0, e_0 \rangle = \langle P(v) e_0, e_0 \rangle = \langle e^{X_v} P_U e^{-X_v} e_0, e_0 \rangle = \langle P_U e^{-X_v} e_0, e^{-X_v} e_0 \rangle \quad (3.35)$$

$$= \langle P_U (1 - X_v + \frac{1}{2} X_u^2 + \dots) e_0, (1 - X_v + \frac{1}{2} X_v^2 + \dots) e_0 \rangle \quad (3.36)$$

$$= \langle P_U e_0 - P_U X_v e_0 + P_U \frac{1}{2} X_u^2 e_0 + \dots, e_0 - X_v e_0 + \frac{1}{2} X_v^2 e_0 + \dots \rangle \quad (3.37)$$

$$= \langle P_U X_v e_0, X_v e_0 \rangle + \dots = |v|^2 + \dots \quad (3.38)$$

where we again use that $X_v e_0 = v$, $P_U v = v$ and $P_U e_0 = 0$. Since we then have $f(\Phi(tv)) = \langle P_U X_{tv} e_0, X_{tv} e_0 \rangle + \dots = t^2 \langle P_U X_v e_0, X_v e_0 \rangle + \dots = t^2 |v|^2 + \dots$ where all the later terms have $t$ to a higher degree, we conclude that

$$\frac{d^2 f(\Phi(tv))}{dt^2}|_{t=0} = 2|v|^2 \quad (3.39)$$

thus $0 \in U$ is a non-degenerate minimum of $f \circ \Phi : U \to \mathbb{R}$. Furthermore, since $\dim_{\mathbb{C}} U = k = codim_{\mathbb{C}}(S_0)$, we infer that $S_0$ is indeed a non-degenerate critical manifold as required.

The proof of $S_1$ being a non-degenerate critical manifold is similar. Let $U \in S_1 = Gr_{k-1}(V)$ be given and denote by $U'$ the orthogonal complement of $U$ in $V$. Then we have

$$\dim_{\mathbb{C}} U' = n - (k-1) = n - k + 1 = codim_{\mathbb{C}} S_1 \quad (3.40)$$

As before, we aim to show that the smooth map

$$\Phi : \mathbb{C} \supset U' \to G_k(W), \qquad \Phi(v) = e^{X_v} U \quad (3.41)$$

is an immersion at $0 \in U'$ and that the composition $f \circ \Phi$ has a non-degenerate maximum at 0. Again we have $P(u) = P_{\Phi(v)}$ and thus $\dot{P}_v = \frac{dP(tv)}{dt}|_{t=0} = [X_v, P_U]$ which gives:

$$\langle \dot{P}_v e_0, v \rangle = \langle X_v P_U e_0 - P_U X_v e_0, v \rangle = \langle v - P_U v, v \rangle = |v|^2 \quad (3.42)$$

since $v \in U'$. Thus $\Phi$ is an immersion at $v = 0$ by the same argument as before. We also have

$$f(\Phi(v)) = \langle P_U e^{-X_v} e_0, e^{-X_v} e_0 \rangle = \langle P_U (1 - X_v + \frac{1}{2} X_v^2 + \dots) e_0, (1 - X_v + \frac{1}{2} X_v^2 + \dots) e_0 \rangle$$

$$= \langle P_U e_0 - P_U X_v e_0 + P_U \frac{1}{2} X_v^2 e_0 + \dots, e_0 - X_v e_0 + \frac{1}{2} X_v^2 e_0 + \dots \rangle \quad (3.43)$$

$$= \langle e_0 + \frac{1}{2} X_v^2 e_0 + \dots, e_0 - v + \frac{1}{2} X_v^2 e_0 - \dots \rangle \quad (3.44)$$

$$= |e_0|^2 + \frac{1}{2} \langle X_v^2 e_0, e_0 \rangle + \frac{1}{2} \langle e_0, X_v^2 e_0 \rangle + \dots = 1 - \langle X_v e_0, X_v e_0 \rangle + \dots \quad (3.45)$$

$$= 1 - |v|^2 + \dots \quad (3.46)$$

Therefore we have $\frac{d^2 f(\Phi(tv))}{dt^2}|_{t=0} = 1 - 2|v|^2$ and from the same arguments as before, we conclude that $S_1$ is a non-degenerate critical manifold as desired. □

Now, by [Morse-Bott lemma], the negative normal bundles are $\nu^- N_0 = 0$ and $\nu^- N_1 = \nu N_1$ thus

$$\lambda(N_0) = 0, \qquad \lambda(N_1) = 2(n - k + 1) \quad (3.47)$$

Furthermore, since in particular $\nu^- N_i$ is a complex vector bundle for $i = 0, 1$, it is orientable, cf. [Lemma 2.2.1]. We conclude that $f$ is an orientable Morse-Bott function as desired. □

Although this proof is a bit more cumbersome than one could have hoped for, this example should make the reader thankful for the time we spent on developing the more potent Morse-Bott theory. We are now finally ready to prove the desired [Proposition 3.1.2] stated above.

*Proof of Proposition 3.1.2.* We carry out a strong induction on $\omega = k + n$. For $\omega = 2$, that is $(k, n) = (1, 1)$, we have $P_{1,1}(t) = 1$ which is then even. Furthermore, $P_{0,1}(t) = 1$ thus we get:

$$P_{1,2}(t) = \sum_{\lambda} B_\lambda(G_{1,2})t^\lambda = 1 + t^2 = P_{1,1}(t) + t^2 P_{0,1}(t) \tag{3.48}$$

where we have used [Example 1.3.1] to conclude that $B_0(G_{1,2}) = B_2(G_{1,2}) = 1$ since $G_{1,2} = \mathbb{C}P^2$. Thereby, the induction start is verified.

Assume that the induction hypothesis holds for $\omega = k + n$, we must prove that it holds for $\omega = k + n + 1$. From the proof of [Lemma 3.1.2] we have $N_0 \simeq G_{k,n}$ and $N_1 \simeq G_{k-1,n}$, which by the induction hypothesis then have corresponding even Poincaré polynomials $P_{N_i}(t)$. By [Corollary 2.2.2], we conclude that $f$ is, in fact, a *perfect* Morse-Bott function. The induction hypothesis also tells us that all Poincaré polynomials $P_{k,n}(t)$ with $k + n \le \omega$ are even. By [Lemma 3.1.2], the map $f : Gr_k(W) \to \mathbb{R}$ is a perfect Morse-Bott function, thus by definition we have

$$P_{k,n+1}(t) = P_f(t) = P_{N_0}(t) + t^{2(n-k+1)}P_{N_1}(t) = P_{k,n}(t) + t^{2(n-k+1)}P_{k-1,n}(t) \tag{3.49}$$

which is then even as well. The desired result now follows by the principle of simple mathematical induction. □

**Corollary 3.1.1.** *The Poincaré polynomial of the complex Grassmannian, $G_{k,n}$, is*

$$P_{k,n}(t) = \frac{\prod_{i=1}^{n}(1 - t^{2i})}{\prod_{j=1}^{k}(1 - t^{2j})\prod_{i=1}^{n-k}(1 - t^{2i})} \tag{3.50}$$

*Proof.* From [Proposition 3.1.2] we have the equation $P_{k,n+1}(t) = P_{k,n}(t) + t^{2(n+1-k)}P_{k-1,n}(t)$. If we make a change in variables, putting $Q_{k,l} = P_{k,n}$ where $l = n - k$, we can write this equation as

$$\begin{aligned} Q_{k,l+1}(t) = Q_{k,n-k+1}(t) &= Q_{k,l}(t) + t^{2(n+1-k)}Q_{k-1,n-(k-1)}(t) \\ &= Q_{k,l}(t) + t^{2(l+1)}Q_{k-1,l+1}(t) \end{aligned} \tag{3.51}$$

Note that we have $Q_{l,k} = P_{l,2k} = P_{n-k,2k} = P_{2k-n+k,2k} = P_{3k-n,2k} = P_{k,n} = Q_{k,l}$, where we have used the diffeomorphism from [Proposition 3.1.1] and the relation $Q_{k,l} = P_{k,n}$ with $l = n - k$. Therefore, we also have, when inserting in (3.51):

$$Q_{k,l+1}(t) = Q_{l+1,k}(t) = Q_{l+1,k-1} + t^{2k}Q_{l,k}(t) = Q_{k-1,l+1} + t^{2k}Q_{k,l}(t) \tag{3.52}$$

Comparing (3.51) and (3.52) gives us the equation

$$(1 - t^{2k})Q_{k,l}(t) = (1 - t^{2(l+1)})Q_{k-1,l+1}(t) \qquad \text{that is} \qquad Q_{k,l}(t) = \frac{(1 - t^{2(l+1)})}{(1 - t^{2k})}Q_{k-1,l+1}(t)$$

Changing back the variables gives us the relation

$$P_{k,n}(t) = \frac{(1 - t^{2(n-k+1)})}{(1 - t^{2k})}P_{k-1,n}(t) \tag{3.53}$$

We see that by iterating this procedure, for $P_{k-1,n}, P_{k-2,n} \dots P_{1,n}$ we obtain

$$P_{k,n}(t) = \frac{(1 - t^{2(n-k+1)})}{(1 - t^{2k})}\frac{(1 - t^{2(n-k+2)})}{(1 - t^{2(k-1)})} \cdots \frac{(1 - t^{2n})}{(1 - t^2)} = \frac{\prod_{i=n-k+1}^{n}(1 - t^{2i})}{\prod_{i=1}^{k}(1 - t^{2i})} \tag{3.54}$$

which we can also write as

$$P_{k,n}(t) = \frac{\prod_{i=1}^{n}(1 - t^{2i})}{\prod_{i=1}^{k}(1 - t^{2i})\prod_{i=1}^{n-k}(1 - t^{2i})} \tag{3.55}$$

thus obtaining the desired expression. □

**Example 3.1.1.** As an example of the efficiency of this result, let us re-discover the homology of $\mathbb{C}P^n = Gr_1(\mathbb{C}^n) = G_{1,n}$. By [Corollary 3.1.1] we obtain

$$P_{1,n} = \frac{\prod_{i=1}^{n}(1 - t^{2i})}{(1 - t^2)\prod_{i=1}^{n-1}(1 - t^{2i})} = \frac{1 - t^{2n}}{1 - t^2} = 1 + t^2 + t^4 + \dots + t^{2(n-1)} \tag{3.56}$$

We see directly from this that $\mathbb{C}P^n$ can be given a CW-complex structure consisting of one cell of each even dimension. By cellular homology, it follows that $H_i(\mathbb{C}P^n) = \mathbb{Z}$ for $i$ even and $H_i(\mathbb{C}P^n) = 0$ for $i$ odd.

## 3.2 Further developments

We have now exhausted just about all we can say about the topology of complex Grassmannians with our rather naïve approach taken above. It is however possible to obtain more information, for the price of introducing the Schubert calculus, the Plücker embedding and analyzing the integral curves of $-\nabla f$ for a perfect Morse-Bott function $f$, which one can actually calculate explicitly in this case, to obtain information about the stable and unstable manifolds, see [Gue02]. This is, however, outside the scope of this project, and we will therefore desist from becoming absorbed in this. Instead, we give a brief discussion on where one could go from here.

### 3.2.1 A route to cohomology

We would like to determine the cohomology of complex Grassmannians, and not just the homology groups. However, it is known that in general, the classical Morse inequalities only give information about the additive structure of the cohomology ring. Nevertheless, it is actually, in this special case, possible to determine the cohomology ring $H^*(Gr_k(\mathbb{C}); \mathbb{Z})$, which has the form

$$\frac{\mathbb{Z}[c_1, \dots, c_{n-k}, d_1, \dots, d_k]}{(1 + c_1 + \cdots + c_{n-k})(1 + d_1 + \cdots + d_k) = 1} \tag{3.57}$$

for $c_1, d_i \in H^{2i}(Gr_k(\mathbb{C}^n); \mathbb{Z})$, by considering the stable and unstable manifolds for a certain Morse function on $Gr_k(\mathbb{C}^n)$, see [Gue02].

One would of course like to know whether it is possible to determine the cohomology of *any* compact manifold $M$. This is related to the question of how the flow lines on a given manifold can be configured. Indeed, just as we have seen that the torus $T^2$ does not allow a Morse function with only three critical points, it is not possible to have arbitrary configurations of flow lines going between the critical points. Therefore, one has to study exactly how these are constrained, and it will be necessary to consider, not just one, but multiple Morse functions on a given manifold. This has in turn led to yet another generalization of Morse theory, from the 90's onwards, in which one can keep the complex Grassmannians in mind as a typical example, cf. [Gue02].

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