More on Curvature and Torsion

If we are going to get far past the helix, we need to learn to compute the Frenet apparatus <u>without</u> an arclength parametrization.

So let $\alpha(t)$ be a curve and ~~$\beta(t(s))$~~ $\beta(s)$ be it's arclength reparametrization.

$$\alpha'(t) = \frac{d}{dt}\beta(s(t)) = \beta'(s(t))\, s'(t)$$

$$= \left(\frac{d}{ds}\beta\right) \cdot s'(t)$$

$$= T(t) \cdot |\alpha'(t)|$$

So we can always find the unit tangent vector $T(t)$ by normalizing $\alpha'(t)$. Let's call $|\alpha'(t)| = v(t)$ for velocity for mow.

Now if we differentiate again,

$$\alpha''(t) = v'(t)\,T(t) + v(t)\,T'(s(t))\,s'(t)$$

$$= v'(t)\,T(t) + v^2(t)\big(\kappa(t)\,N(t)\big)$$

and so if we cross with $\alpha'(t)$, we get

$$\alpha'(t) \times \alpha''(t) = vT \times \left(v'T + v^2\kappa N\right)$$

$$= v^3 \kappa N$$

and

$$|\alpha'(t) \times \alpha''(t)| = |v^3|\,\kappa.$$

$\kappa \geq 0$, remember!

so

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|v(t)|^3} = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3}.$$

We can now do some examples.

Example. Find the curvature of the tractrix

$$\alpha(\theta) = (\cos\theta + \ln\tan\theta/2, \sin\theta)$$

We compute

$$\alpha'(\theta) = \left(-\sin\theta + \frac{d}{d\theta}\ln\frac{\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}}, \cos\theta\right)$$

$$= \left(-\sin\theta + \frac{1}{2}\left(\frac{d}{d\theta}\left[\ln(1-\cos\theta) - \ln(1+\cos\theta)\right]\right), \cos\theta\right)$$

$$= \left(-\sin\theta + \frac{\sin\theta}{2(1-\cos\theta)} + \frac{\sin\theta}{2(1+\cos\theta)}, \cos\theta\right)$$

$$= \left(-\sin\theta + \frac{\sin\theta}{2}\left(\frac{(1+\cancel{\cos\theta}) + (1-\cancel{\cos\theta})}{1-\cos^2\theta}\right), \cos\theta\right)$$

$$= \left(-\sin\theta + \frac{\cancel{2}\,\cancel{\sin\theta}}{\cancel{2}\sin^{\cancel{2}}\theta}, \cos\theta\right)$$

$$= (-\sin\theta + \csc\theta, \cos\theta)$$

so

$$|\alpha'(\theta)|^2 = (-\sin\theta + \csc\theta)^2 + \cos^2\theta$$

$$= \cancel{\sin^2\theta} - 2\sin\theta \underset{\uparrow 1/\sin\theta}{\csc\theta} + \csc^2\theta + \cancel{\cos^2\theta}$$

$$= 1 - 2 + \csc^2\theta$$

$$= \csc^2\theta - 1 \quad \leftarrow \text{recall } \sin^2\theta + \cos^2\theta = 1, \quad 1 + \cot^2\theta = \csc^2\theta$$

$$= \cot^2\theta$$

which means that

because $\theta \in [\pi/2, \infty)$, $\cot\theta$ is <u>negative</u>.

$$v(t) = |\cot\theta| = -\cot\theta$$

Now

$$\alpha''(\theta) = (-\cos\theta - \csc\theta\cot\theta, -\sin\theta)$$

so the only term in the cross product that's nonzero is the third:

$$(-\sin\theta + \csc\theta)(-\sin\theta) - \cos\theta(-\cos\theta - \csc\theta\cot\theta)$$

$$= \cancel{\sin^2\theta} - \cancel{1} + \cancel{\cos^2\theta} + \csc\theta\cot\theta\cos\theta = \cot^2\theta.$$

We then compute

$$\kappa(t) = \frac{|\alpha'(t) \times \alpha''(t)|}{|\alpha'(t)|^3} = \frac{\cancel{\cot^2\theta}}{-\cot^{\cancel{3}}\theta} = -\tan\theta.$$

The general formula for torsion is basically similar to work out; you'll prove for homework that it's

$$\tau = \frac{\alpha' \cdot (\alpha'' \times \alpha''')}{|\alpha' \times \alpha''|^2}.$$

Example. Compute κ and τ for $\gamma(t) = (t, t^2, t^3)$.

We find

$$\gamma'(t) = (1, 2t, 3t^2)$$
$$\gamma''(t) = (0, 2, 6t)$$

and

$$\gamma'(t) \times \gamma''(t) = (12t^2 - 6t^2, 6t - 0, 2 - 0)$$
$$= (6t^2, 6t, 2)$$

It is now easy to work out

$$\kappa(t) = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}$$

To work out torsion, it's helpful to use a property of the triple product:

$$\alpha' \cdot (\alpha'' \times \alpha''') = \alpha''' \cdot (\alpha' \times \alpha'')$$

Now

$$\gamma'''(t) = (0, 0, 6),$$

so

$$\tau(t) = \frac{12}{36t^4 + 36t^2 + 4}. \qquad \square$$

Why are κ and τ important?

Proposition. κ and τ are invariant when γ is transformed by a rigid motion (translation + rotation).

~~Proof~~. This is a special case of a theorem about framed curves

Theorem. If γ is framed by F, and A is in $SO(3)$, then $A\gamma$ is framed by AF. Further, if $F' = FS$, $(AF)' = AFS$, where S is skew-symmetric. (Thus the coefficient functions of S, α_{12}, α_{13}, and α_{23} are invariant under rotations of γ.)

This means that κ, τ give us a way to talk about properties of γ invariant under rigid motions - these properties are called <u>geometric</u>.

Proposition. A space curve is a line $\Leftrightarrow$ its curvature $\kappa(s) \equiv 0$.

Proof. ($\Rightarrow$) Since $\gamma(s) = s\vec{v} + \vec{c}$, $T(s) = \vec{v}$, $T'(s) = 0$, and $\kappa \equiv 0$.

($\Leftarrow$) Since $\kappa \equiv 0$, $T'(s) = 0$, and $T(s) = \vec{v}$ for some fixed $\vec{v}$. But then $\gamma'(s) = \vec{v}$. and integrating yields $\gamma(s) = \vec{v}s + \vec{c}$. $\square$

It's a little harder to prove the "lock-on theorem":

Proposition. If all tangent lines of a curve $\gamma(s)$ pass through $\vec{0}$, then γ is a line through $\vec{0}$.

Proof. By hypothesis, $\exists$ some scalar function $\lambda(s)$ so $\gamma(s) + \lambda(s)T(s) = 0$.

~~So~~ or

$$\gamma(s) = -\lambda(s)\,T(s)$$

Differentiating,

$$\gamma'(s) = -\lambda'(s)\,T(s) + \lambda(s)\,\kappa(s)\,N(s),$$

or

$$T(s) = -\lambda'(s)\,T(s) + \lambda(s)\,\kappa(s)\,N(s).$$

Here's a neat trick. We can rewrite this as

$$(1+\lambda'(s))\,T(s) = \lambda(s)\,\kappa(s)\,N(s).$$

But since $T(s)$ and $N(s)$ are orthogonal, this must mean that

$$1+\lambda'(s) = 0 \quad \text{and} \quad \lambda(s)\,\kappa(s) = 0.$$

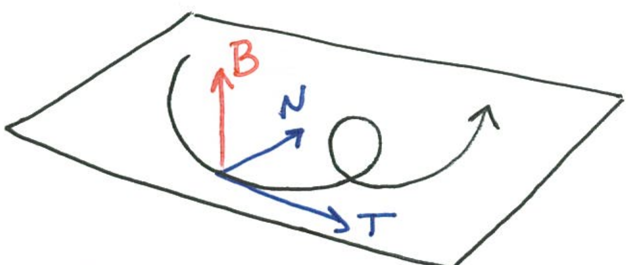
The left equation $\Rightarrow$ $\lambda' \equiv -1$, so $\lambda(s) = c - s$ for some c. Then the right equation $\Rightarrow$ $\kappa \equiv 0$. Thus γ is a line, which passes through the origin when $s = c$. $\square$

We now prove something harder.

Proposition. A space curve γ with nonvanishing κ is planar $\Leftrightarrow \tau \equiv 0$.

Wlog, we can assume $\gamma(0) = \vec{0}$ and that γ is parametrized by arclength.

Proof ($\Rightarrow$) If γ is contained in a plane P, at each s, $T(s), N(s)$ are in P. Thus, $T(s) \times N(s)$ is the normal vector to P.



Since this normal is constant

$$B'(s) = -\tau(s) N(s) = 0,$$

and torsion must be zero.

($\Leftarrow$) If $\tau(s) \equiv 0$, $B(s)$ is a constant B_0. Consider $f(s) = \langle \gamma(s), B_0 \rangle$. At $s=0$, $f(0)=0$. But

$$f'(s) = \langle T(s), B_0 \rangle = \langle T(s), B(s) \rangle = 0,$$

so $f(s) \equiv 0$ and $\gamma(s)$ is in the plane normal to B_0. $\square$

Let's review:

$\kappa = 0 \implies$ straight line

$\tau = 0 \implies$ planar

$\kappa = c_1, \tau = c_2 \implies$ helix (homework).

This seems to imply that fixing "one or both of" κ and τ makes the curve very special. This intuition is strengthened by

Proposition. $\kappa(s) \neq 0$ and All tangent vectors of $\gamma(s)$ make a constant angle with some fixed $\vec{v}$ $\iff$ τ/κ is a constant.

A curve like this is called a <u>generalized helix</u>.

Proof. ($\Rightarrow$) We know $\langle T(s), \vec{v} \rangle = \cos\theta =$ constant, so (differentiating),

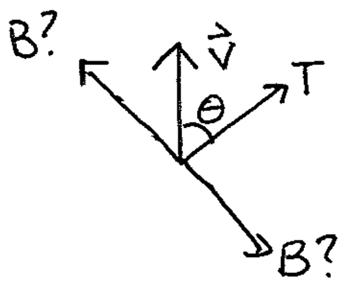
$$\langle \kappa(s) N(s), \vec{v} \rangle = 0$$

Since $\kappa(s) \neq 0$, this implies $\langle N(s), \vec{v} \rangle = 0$.

Now, differentiating again,

$$\langle -\kappa(s) T(s) + \tau(s) B(s), \vec{v} \rangle = 0$$

Since $\langle N(s), \vec{v} \rangle = 0$, $\vec{v}$ is in the T, B plane. We know $\langle T(s), \vec{v} \rangle = \cos\theta$, so $\langle T(s), B(s) \rangle = 0 \Rightarrow \langle B(s), \vec{v} \rangle = -\sin\theta$.



$= \cos(\frac{\pi}{2}+\theta)$

or

$= +\sin\theta$

$= \cos(\frac{\pi}{2}-\theta)$

Thus

$$-\kappa(s)\cos\theta \pm \tau(s)\sin\theta = 0$$

and

$$\frac{\tau(s)}{\kappa(s)} = \pm\frac{\cos\theta}{\sin\theta} = \pm\cot\theta, \text{ which is constant!}$$

($\Leftarrow$) Given that $\frac{\tau(s)}{\kappa(s)}$ is constant, let it equal $\cot\theta$, and set

$$\vec{v}(s) = \cos\theta\, T(s) + \sin\theta\, B(s)$$

We'll then compute

$$\vec{v}\,'(s) = (\kappa(s)\cos\theta - \tau(s)\sin\theta)\,N(s)$$

$$= 0$$

work it out, but comes from cross multiplying in

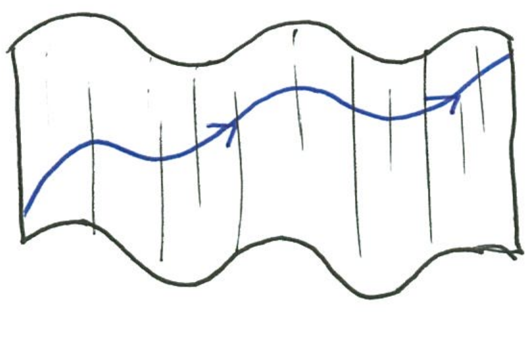
$$\frac{\tau(s)}{\kappa(s)} = \frac{\cos(\theta)}{\sin(\theta)}.$$

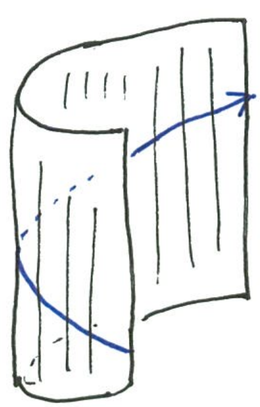
so $\vec{v}$ is a constant vector. But

$\langle \vec{v}, T(s) \rangle = \cos\theta$, which is constant. $\square$

Note that we got ~~used~~ $\kappa \neq 0$ in the $(\Leftarrow)$ part of the proof "for free" because the ratio $\tau(s)/\kappa(s)$ existed.

A generalized helix actually lies on a flat surface formed by extending the $\vec{v}$ direction.





So what if we fix

$$\kappa(s) = c \qquad \underline{\text{or}} \qquad \tau(s) = c$$

and let the other function vary as you like? Do we learn anything about the curve? Very little!

Theorem [Ghomi, 2006]

If γ is a curve of maximum curvature K and $K_2 \geq K$, then for any $\epsilon > 0$ $\exists$ a curve γ_2 of constant curvature K_2 so that

$$|\gamma(s) - \gamma_2(s)| < \epsilon \quad \underline{\text{and}} \quad |\gamma'(s) - \gamma_2(s)| < \epsilon$$

for all s.

A similar statement holds for curves of constant torsion.