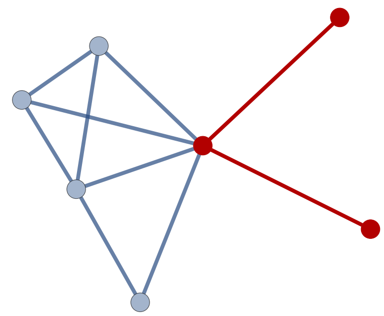
MATH/CSCI 4690/6690 : Eigenvalues and eigenvectors of example graphs.

In this homework, we'll work out some computations of eigenvalues and eigenvectors which support our notes on the zoo of graphs. The purpose of our work is to practice the techniques we've built up for analyzing eigenvalues and eigenvectors of graphs.

1. (10 points) Suppose that we have two vertices a and b of degree 1 in a graph $\mathbf{G}$ which are both joined to a common vertex c, as in the picture below:



Prove that the vector $\vec{x} = \delta(a) - \delta(b)$[a] is an eigenvector of $L_{\mathbf{G}}$ with eigenvalue 1.

Hint: We know that

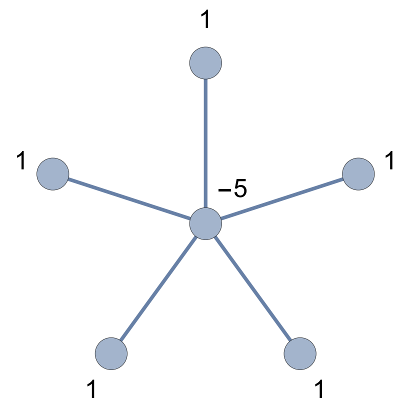
$$L_{\mathbf{G}}(\vec{x})(v) = \sum_{w \text{ with } v \bullet\!\!-\!\!\bullet w} \vec{x}(v) - \vec{x}(w).$$

[a] That is, the vector with $\vec{x}(a) = +1$, $\vec{x}(b) = -1$, and $\vec{x}(-) = 0$ for all other vertices.

2. (20 points) Suppose $\mathbf{S_v}$ is the star graph with $\mathbf{v}$ vertices[b] and

$$\vec{x}(v_i) = \begin{cases} -(\mathbf{v}-1), & \text{if } i = 1, \\ 1, & \text{otherwise} \end{cases}$$

as shown in



We proved in the notes that $\vec{x}$ is an eigenvector of $\mathbf{S_v}$.

(1) (10 points) Prove that $\vec{x}$ has eigenvalue $\mathbf{v}$ by computing the Rayleigh quotient

$$\frac{\langle \vec{x}, \vec{x} \rangle_{\mathbf{S_v}}}{\langle \vec{x}, \vec{x} \rangle}$$

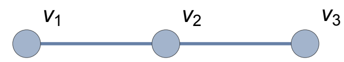
[b] Recall that this is the $\mathbf{v}$ vertex graph with edges are $v_1 \bullet\!\!-\!\!\bullet\, v_2, v_1 \bullet\!\!-\!\!\bullet\, v_3, \ldots, v_1 \bullet\!\!-\!\!\bullet\, v_{\mathbf{v}}$.

(2) (10 points) Now use

$$L_{\mathbf{G}}(\vec{x})(v) = \sum_{w \text{ with } v \bullet\!\!-\!\!\bullet w} (\vec{x}(v) - \vec{x}(w)).$$

to show directly both that $\vec{x}$ is an eigenvector and that its eigenvalue is $\mathbf{v}$.

3. (10 points) Consider the path graphs $\mathbf{G}$ and $\mathbf{H}$



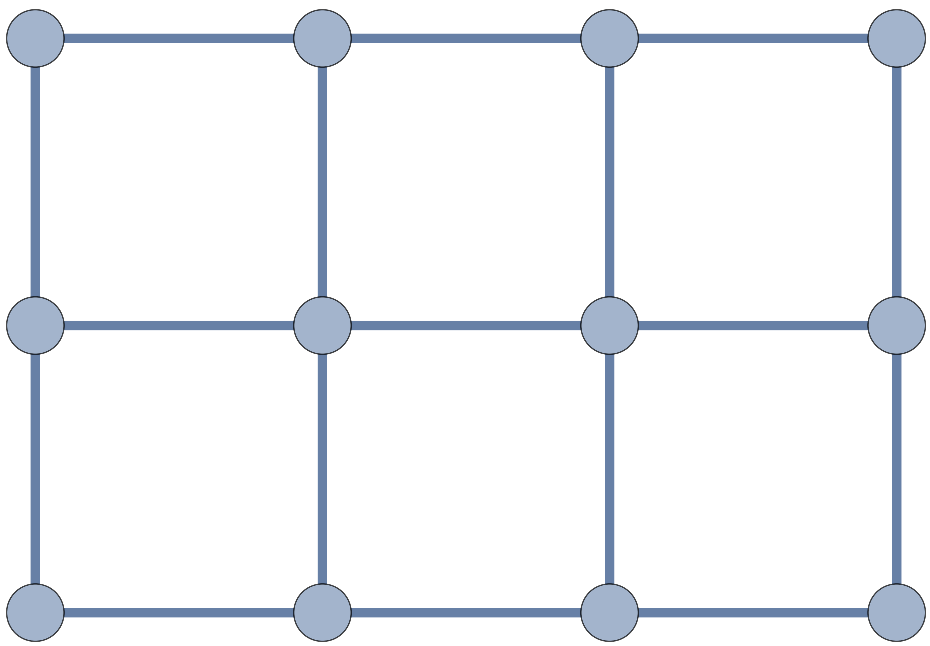
Definition. *The graph product of graphs* $\mathbf{G}$ *and* $\mathbf{H}$ *with vertex sets* $V_{\mathbf{G}}$ *and* $V_{\mathbf{H}}$ *is the graph* $\mathbf{G} \times \mathbf{H}$ *with vertex set*

$$V_{\mathbf{G}\times\mathbf{H}} = \{(a, b) \mid a \in V_{\mathbf{G}} \wedge {}^{c} b \in V_{\mathbf{H}}\}$$

and edge set

$$E_{\mathbf{G}\times\mathbf{H}} = \{(a, b) \bullet\!\!-\!\!\bullet (\hat{a}, \hat{b}) \mid (a \bullet\!\!-\!\!\bullet \hat{a} \wedge b = \hat{b}) \vee (a = \hat{a} \wedge b \bullet\!\!-\!\!\bullet \hat{b})\} \tag{1}$$

Prove that the graph product of the path graphs above is the grid graph $\mathbf{G} \times \mathbf{H}$



by labeling each vertex of the grid graph with the corresponding pair of vertices in $\mathbf{G}$ and $\mathbf{H}$ and identifying each edge of the grid graph with an element of the set $E_{\mathbf{G}\times\mathbf{H}}$ given in (1).

[c] The symbol $\wedge$ is the "logical and", while $\vee$ is the "logical or". We'll use both to simplify our definitions.

4. (20 points) Recall that

Definition. *The hypercube H_d of dimension d is the graph with vertex set*[d]

$$V_{H_d} = \{0,1\}^d = \{(b_1,\ldots,b_d) \mid b_i = 0 \vee b_i = 1\}$$

where

$$E_{H_d}\{(b_1,\ldots,b_d) \bullet\!\!-\!\!\bullet (c_1,\ldots,c_d) | \; b_i \neq c_i \textit{ for exactly one } i \in \{1,\ldots,d\}\}$$

Prove that the 2^d eigenvectors of L_{H_d} may each be identified with a string $y = (y_1,\ldots,y_d) \in \{0,1\}^n$ so that at each vertex $x = (x_1,\ldots,x_d) \in V_{H_d}$,

$$\vec{\psi}_y(x) = (-1)^{\sum y_i x_i}$$

Hint: Proceed by induction on d. Use the fact that $H_d = H_{d-1} \times H_1$ and prove that the eigenvector $\psi_{(y_1,\ldots,y_{d-1})}$ of H_{d-1} leads to the eigenvectors $\psi_{(y_1,\ldots,y_{d-1},0)}$ and $\psi_{(y_1,\ldots,y_{d-1},1)}$ of H_d.

[d]Notice that H_d has the same vertices as the complete binary tree T_d, but a very different edge set!

5. (60 points)

Definition. *If A is an $m \times n$ matrix and B is a $p \times q$ matrix, the* Kronecker product $A \otimes B$ *of A and B is the $mp \times nq$ block matrix given by*

$$A \otimes B = \begin{pmatrix} a_{11}B & \cdots & a_{n1}B \\ \vdots & \ddots & \vdots \\ a_{1m}B & \cdots & a_{mn}B \end{pmatrix}$$

This definition is precise, but it doesn't give much motivation for the Kronecker product. So to see where this comes from, let's consider two different ways to combine vector spaces:

Definition. *Given finite-dimensional vector spaces V and W with dimensions m and n and bases $\vec{v}_1, \ldots, \vec{v}_m$ and $\vec{w}_1, \ldots, \vec{w}_n$, we define the* direct product *of the vector spaces (also called the* Cartesian product *or simply the* product*):*

$$V \times W = \text{ the } m + n \text{ dimensional vector space with basis } \vec{v}_1, \ldots, \vec{v}_m, \vec{w}_1, \ldots, \vec{w}_n$$

Equivalently, we can write

$$V \times W := \{(\vec{v}, \vec{w}) \mid \vec{v} \in V \wedge \vec{w} \in W\}$$

with the scalar multiplication and sum operations

$$\lambda(\vec{v}, \vec{w}) = (\lambda\vec{v}, \lambda\vec{w})$$
$$(\vec{v}, \vec{w}) + (\vec{p}, \vec{q}) = (\vec{v} + \vec{p}, \vec{w} + \vec{q})$$

We can also write a vector in $V \times W$ as an $(m + n) \times 1$ column vector

$$(\vec{v}, \vec{w}) = \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix}$$

If we have linear maps $A\colon V_1 \to V_2$ and $B\colon W_1 \to W_2$, we can define $A \times B\colon V_1 \times W_2 \to V_2 \times W_2$ by $(A \times B)(\vec{v}, \vec{w}) = (A\vec{v}, B\vec{w})$. As a matrix,

$$A \times B = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

so that

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \vec{v} \\ \vec{w} \end{bmatrix} = \begin{bmatrix} A\vec{v} \\ B\vec{w} \end{bmatrix}$$

Another option for combining vector spaces is

Definition. *Given finite-dimensional real vector spaces V and W with dimensions m and n and bases $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{w}_1, \dots, \vec{w}_n$, we define the* tensor product *of the vector spaces:*

$$V \otimes W = \text{ the } mn \text{ dimensional vector space with corresponding basis} \\ \vec{v}_1 \otimes \vec{w}_1, \vec{v}_1 \otimes \vec{w}_2, \dots, \vec{v}_1 \otimes \vec{w}_n, \vec{v}_2 \otimes \vec{w}_1, \dots, \vec{v}_i \otimes \vec{w}_j, \dots, \vec{v}_m \otimes \vec{w}_n.$$

Note that the ordering of the corresponding basis for $V \otimes W$ is part of the definition.

Here the notation means $\vec{v}_i \otimes \vec{w}_j$ means "the new basis element obtained by pairing $\vec{v}_i$ and $\vec{w}_j$" and nothing more.

Proposition. *Given finite-dimensional (real)*[e] *vector spaces V and W with dimensions m and n and bases $\vec{v}_1, \dots, \vec{v}_m$ and $\vec{w}_1, \dots, \vec{w}_n$, let $\Delta(i, j)$ be the matrix with 1 in the i, j position and zeros elsewhere. The linear map defined by $\vec{v}_i \otimes \vec{w}_j \mapsto \Delta(i, j)$ is an isomorphism between the tensor product $\vec{V} \otimes \vec{W}$ and the vector space*[f] $\mathrm{Mat}_{m \times n}$ *of (real) $m \times n$ matrices.*[g]

Since we have this proposition, it is often useful to blur the distinction between $\vec{V} \otimes \vec{W}$ and $\mathrm{Mat}_{m \times n}$ when writing about tensor products, in the same way that we often blur the distinction between an m-dimensional real vector space V and $\mathbb{R}^m$ because V is isomorphic to $\mathbb{R}^m$ as a vector space and it's easier to discuss m-tuples of real numbers than arbitrary linear combinations of m basis vectors. With this convention ($V \otimes W \cong \mathrm{Mat}_{m \times n}$) established,

Definition. *There is a bilinear map $V \times W \to V \otimes W$ given by*

$$(\vec{v}, \vec{w}) \mapsto \vec{v}\vec{w}^T := \vec{v} \otimes \vec{w}$$

where we use the $m \times n$ matrix $\vec{v}\vec{w}^T$ to define the element $\vec{v} \otimes \vec{w}$ of $V \otimes W$.

Of course, we expect from counting dimensions that the map $V \times W \to V \otimes W$ can't be surjective, since $\dim(V \times W) = m + n$ and $\dim(V \otimes W) = mn$, and it's true that the "pure" or "simple" or "rank-one" tensors $\vec{v} \otimes \vec{w}$ are only a small part[h] of the vector space $V \otimes W$.

[e] From now on, I'm just going to assume that we're discussing real vector spaces and matrices.

[f] It's not hard to convince yourself that matrices can be added and scalar-multiplied, so they must be a vector space.

[g] You won't go far wrong by thinking "a tensor product of two vector spaces is a vector space of matrices".

[h] If you want to know *which* elements of $V \otimes W$ are simple tensors, here's a short form of that story. First, it's a common mistake to assume that the simple tensors are a linear subspace of $V \otimes W$ (they aren't) and be confused by the fact that there's a basis for $V \otimes W$ composed of simple tensors (true) but not every element of $V \otimes W$ is a simple tensor (also true). Try to avoid this.

Definition. *An element A of $V \otimes W$ is called a* simple tensor *or* rank one *tensor if it can be written as $\vec{v} \otimes \vec{w}$ for some $\vec{v} \in V$ and $\vec{w} \in W$.*

Definition. *The tensor rank k of an element $A \in V \otimes W$ is the smallest number of rank one tensors $\vec{a}_i \otimes \vec{b}_i$ in $V \otimes W$ so that $A = \vec{a}_1 \otimes \vec{b}_1 + \cdots + \vec{a}_k \otimes \vec{b}_k$.*

Proposition. *If $A \in V \otimes W$, the tensor rank of A is equal to the rank of the corresponding matrix $[A] \in \mathrm{Mat}_{m \times n}$.*

This proposition tells us exactly how to understand the simple tensors in $V \otimes W$: they are the rank-1 matrices in $\mathrm{Mat}_{m \times n}(\mathbb{R})$.

Now we get to the Kronecker product.

Definition. *If we have linear maps $A\colon V_1 \to V_2$ and $B\colon W_1 \to W_2$, we can define a corresponding linear map $A \otimes B\colon V_1 \otimes W_1 \to V_2 \otimes W_2$ in two (equivalent) ways:*

1. *Map simple tensors to simple tensors by*

$$(A \otimes B)(\vec{v} \otimes \vec{w}) = (A\vec{v}) \otimes (B\vec{w})$$

   and "extend the definition to higher-rank tensors by linearity".[i]

2. *If $\dim V_i = m_i$ and $\dim W_i = n_i$, define the map $(A \otimes B)\colon \mathrm{Mat}_{m_1 \times n_1} \to \mathrm{Mat}_{m_2 \times n_2}$ by*

$$X \in \mathrm{Mat}_{m_1 \times n_1} \mapsto AXB^T \in \mathrm{Mat}_{m_2 \times n_2}$$

Now $A \otimes B$ is a linear map from an $m_1 n_1$-dimensional vector space $V_1 \otimes W_1$ to an $m_2 n_2$-dimensional vector space $V_2 \otimes W_2$, so we must be able to represent it as an $m_2 n_2 \times m_1 n_1$ matrix. In fact

Proposition. *Suppose V_1, V_2 are vector spaces with dimensions m_1 and m_2, and W_1, W_2 are vector spaces with dimensions n_1 and n_2. Further, suppose we have chosen bases for V_1 and V_2 and W_1 and W_2, and we have corresponding standard bases for $V_1 \otimes W_1$ and $V_2 \otimes W_2$.*

If $A\colon V_1 \to V_2$ and $B\colon W_1 \to W_2$ are linear maps written as $m_2 \times m_1$ and $n_2 \times n_1$ matrices with respect to our bases, then the linear map $A \otimes B\colon V_1 \otimes W_1 \to V_2 \otimes W_2$ is written as an $m_2 n_2 \times m_1 n_1$ matrix[j] *by the Kronecker product $A \otimes B$ of the matrices A and B.*

[i] Here the casual use of "extend to higher-rank tensors by linearity" should make your blood run cold. Not even all of the simple tensors are linearly independent from one another. So it's definitely possible to define a map on simple tensors which can't be linear in the first place, much less have a linear extension to the entire tensor product. The second definition is intended to reassure you that this operation is, in fact, well-defined.

[j] With respect to the corresponding bases on $V_1 \otimes W_1$ and $V_2 \otimes W_2$, remembering the ordering on these bases.

(1) (10 points) Make sure that you understand the definition by writing out $A \otimes B$ and $B \otimes A$ where

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad B = \begin{pmatrix} 6 \\ 7 \end{pmatrix}$$

Does $A \otimes B = B \otimes A$?

(2) (20 points)

Proposition. *The Kronecker product is bilinear (over matrix addition and scalar multiplication) and associative*

$$\begin{aligned} A \otimes (B + C) &= A \otimes B + A \otimes C \\ (A + B) \otimes C &= A \otimes C + B \otimes C \\ (\lambda A) \otimes B &= A \otimes (\lambda B) = \lambda(A \otimes B) \\ (A \otimes B) \otimes C &= A \otimes (B \otimes C) \end{aligned}$$

The matrix operations of transpose and inverse distribute over the Kronecker product

$$\begin{aligned} (A \otimes B)^{-1} &= A^{-1} \otimes B^{-1} \\ (A \otimes B)^T &= (A^T) \otimes (B^T) \end{aligned}$$

Proposition. *If A, B, C, D are matrices so that the matrix products AC and BD exist*[k] *then*

$$(A \otimes B)(C \otimes D) = (AC) \otimes (BD).$$

This is called the "mixed product property".

Use these properties to prove the following:

Proposition. *Suppose A is an $n \times n$ matrix with eigenvectors $\vec{\alpha}_1, \ldots, \vec{\alpha}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ and B is an $m \times m$ matrix with eigenvectors $\vec{\beta}_1, \ldots, \vec{\beta}_m$ and eigenvalues $\mu_1, \ldots, \mu_m$. Thinking of the eigenvectors as column matrices, prove that the eigenvectors of $A \otimes B$ are in the form $\vec{\alpha}_i \otimes \vec{\beta}_j$ with corresponding eigenvalues $\lambda_i \mu_j$.*

[k] Matrices of the correct sizes to be multiplied are called *conformable*.

(3) (10 points)

Definition. *Suppose that A and B are square matrices of size $n \times n$ and $m \times m$ and I_k denotes the $k \times k$ identity matrix. The* Kronecker sum *$A \oplus B$ is defined by*

$$A \oplus B = A \otimes I_m + I_n \otimes B$$

Prove the following:

Proposition. *Suppose A is an $n \times n$ matrix with eigenvectors $\vec{\alpha}_1, \ldots, \vec{\alpha}_n$ and corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$ and B is an $m \times m$ matrix with eigenvectors $\vec{\beta}_1, \ldots, \vec{\beta}_m$ and eigenvalues $\mu_1, \ldots, \mu_m$. The eigenvectors of $A \oplus B$ are in the form $\vec{\alpha}_i \otimes \vec{\beta}_i$ with corresponding eigenvalues $\lambda_i + \mu_j$.*

(4) (10 points) If $\mathbf{G}$ and $\mathbf{G}'$ are graphs, prove directly that there is a basis for the vertex space $\mathbb{R}^{\mathbf{v}_{\mathbf{G}\times\mathbf{G}'}}$ in which the graph Laplacian $L_{\mathbf{G}\times\mathbf{H}}$ is given by the Kronecker sum:

$$L_{\mathbf{G}\times\mathbf{G}'} = L_{\mathbf{G}} \oplus L_{\mathbf{G}'}.$$

Hint: $L_{\mathbf{G}}\colon \mathbb{R}^{\mathbf{v}_{\mathbf{G}}} \to \mathbb{R}^{\mathbf{v}_{\mathbf{G}}}$ and $L_{\mathbf{G}}\colon \mathbb{R}^{\mathbf{v}'_{\mathbf{G}}} \to \mathbb{R}^{\mathbf{v}'_{\mathbf{G}}}$; as matrices they are written with respect to the bases $v_1, \ldots, v_{\mathbf{v}_{\mathbf{G}}}$ for $\mathbb{R}^{\mathbf{v}_{\mathbf{G}}}$ and $v'_1, \ldots, v'_{\mathbf{v}'_{\mathbf{G}}}$ for $\mathbb{R}^{\mathbf{v}'_{\mathbf{G}}}$.

The Kronecker sum matrix $L_{\mathbf{G}} \oplus L_{\mathbf{G}'}\colon \mathbb{R}^{\mathbf{v}_{\mathbf{G}}+\mathbf{v}'_{\mathbf{G}}} \to \mathbb{R}^{\mathbf{v}_{\mathbf{G}}+\mathbf{v}'_{\mathbf{G}}}$. The definition of Kronecker sum (and Kronecker product) imply that this is written with respect to a particular basis on $\mathbb{R}^{\mathbf{v}_{\mathbf{G}}+\mathbf{v}'_{\mathbf{G}}}$. Your first step is to find this basis.

On the other hand, the graph Laplacian $L_{\mathbf{G}\times\mathbf{G}'}\colon \mathbb{R}^{\mathbf{v}_{\mathbf{G}\times\mathbf{G}'}} \to \mathbb{R}^{\mathbf{v}_{\mathbf{G}\times\mathbf{G}'}}$ is written with respect to a standard basis given by the vertex set of $\mathbf{G} \times \mathbf{G}'$. The next step is to find a correspondence between this basis and your basis for $\mathbb{R}^{\mathbf{v}_{\mathbf{G}}+\mathbf{v}'_{\mathbf{G}}}$.

The final step is to show that the matrices for $L_{\mathbf{G}\times\mathbf{G}'}$ and $L_{\mathbf{G}} \oplus L_{\mathbf{G}'}$ (with respect to your bases) are equal.

(5) (10 points) Now find the eigenvalues of $\mathbf{G} \times \mathbf{H}$ using the last two parts of this question, thereby providing an alternate (and more conceptual) proof of Theorem 6.3.2 in your book (and the notes).