Math 4250 Minihomework: Understanding Quadratic Surfaces.

Definition. *We say that* $x_1^{p_1} \cdots x_n^{p_n}$ *is a* monomial in $x_1, \dots, x_n$ with degree k if $p_1 + \cdots + p_n = k$. *A linear combination of monomials is called a* polynomial in $x_1, \dots, x_n$ of degree k *if the monomials have maximum degree* k. *If* $\vec{x} = (x_1, \dots, x_n)$ *we will also use the phrases "monomial of degree* k *in* $\vec{x}$*" and "polynomial of degree* k *in* $\vec{x}$*".*

Definition. *A* paraboloid *is the graph of a quadratic*[a] *polynomial* $p\colon \mathbb{R}^n \to \mathbb{R}$.

From the multivariable Taylor theorem, we know that every function[b] is locally approximated by a quadratic polynomial called its Taylor expansion around the point $\vec{x}_0$:

$$f(\vec{x}_0 + \vec{x}) \simeq f(\vec{x}_0) + \langle \vec{x}, \nabla f(\vec{x}_0) \rangle + \frac{1}{2} \langle \vec{x}, \mathcal{H} f(\vec{x}_0) \vec{x} \rangle \qquad (\spadesuit)$$

Therefore, the graph of every function[c] is locally approximated by a paraboloid. For that reason, we devote this minihomework to getting a feel for the geometry of paraboloids.

1. (15 points) Suppose $p(\vec{x})$ is a quadratic polynomial. We can always write $p(\vec{x})$ in the form[d]

$$p(\vec{x}) = \sum_{1 \le i \le n} a_{ii}\, x_i^2 + \sum_{1 \le i < j \le n} 2 a_{ij} x_i x_j + \sum_{1 \le i \le n} b_i\, x_i + c. \qquad (\bigstar)$$

In this problem, we'll define a vector $\vec{b} \in \mathbb{R}^n$ and a symmetric $n \times n$ matrix A by

$$\vec{b} = (b_1, \dots, b_n) \quad \text{and} \quad A_{ij} = \begin{cases} a_{ij}, & \text{if } i \le j \\ a_{ji}, & \text{if } i > j. \end{cases} \qquad (\diamondsuit)$$

(1) (5 points) Prove that

$$p(\vec{x}) = \langle \vec{x}, A\vec{x} \rangle + \left\langle \vec{b}, \vec{x} \right\rangle + c = \sum_{ij} A_{ij} x_i x_j + \sum_i b_i x_i + c.$$

[a] Quadratic means "degree 2" in the sense of the definition above of degree for polynomials in several variables.

[b] Technically, this is only true for smooth enough functions. But we're not trying to be precise yet.

[c] And we will soon see that every surface is locally such a graph!

[d] Be careful to note the inequalities on the indices in the sum. For a two variable polynomial, this form evaluates to $p(x_1, x_2) = a_{11} x_1^2 + a_{22} x_2^2 + 2a_{12} x_1 x_2 + b_1 x_1 + b_2 x_2 + c$.

(2) (5 points) Now show that $\nabla p(\vec{x}) = 2A\vec{x} + \vec{b}$ and $\mathcal{H}p(\vec{x}) = 2A$ (for all $\vec{x}$). It's helpful to compute the general first and second partials $\frac{\partial}{\partial x_k} p(\vec{x})$ and $\frac{\partial^2}{\partial x_l \partial x_k} p(\vec{x})$ to get oriented.

(3) (5 points) Using the first two parts of the problem, prove that the quadratic Taylor approximation of $p(\vec{x})$ around $\vec{x}_0 = 0$ is equal to $p(\vec{x})$ itself.[e]

[e]This is a general fact about polynomials– they are their own Taylor approximations.

2. (20 points) In linear algebra, we learned

Definition. *An orthonormal basis for $\mathbb{R}^n$ is a collection of vectors $\vec{v}_1, \ldots, \vec{v}_n \in \mathbb{R}^n$ with*

$$\langle \vec{v}_i, \vec{v}_j \rangle = \begin{cases} 1, & \text{if } i = j \text{ (the vectors are “normal”ized)} \\ 0, & \text{if } i \neq j \text{ (the vectors are “ortho”gonal).} \end{cases}$$

This is also written as $\langle \vec{v}_i, \vec{v}_j \rangle = \delta_{ij}$ where δ_{ij} is the Kronecker delta.[f]

As you might guess from the names, there's a tight connection between orthonormal bases and orthogonal matrices:

Proposition. *The vectors $\vec{v}_1, \ldots, \vec{v}_n$ form an orthonormal basis for $\mathbb{R}^n \iff$ the matrix V with columns $\vec{v}_1, \ldots, \vec{v}_n$ is an orthogonal matrix.*

Suppose we change our basis for $\mathbb{R}^n$ between the standard basis $\vec{e}_i$ and the new basis $\vec{v}_i$, and

$$x_1\vec{e}_1 + \cdots + x_n\vec{e}_n = u_1\vec{v}_1 + \cdots + u_n\vec{v}_n,$$

are two different ways to write the same vector (in the two different bases). If $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{u} = (u_1, \ldots, u_n)$, then

$$\vec{x} = V\vec{u} \quad \text{and} \quad \vec{u} = V^T\vec{x} \tag{$\heartsuit$}$$

Therefore, to rewrite a function $f(x_1, \ldots, x_n)$ as a function $g(u_1, \ldots, u_n)$, we define

$$g(\vec{u}) = f(\vec{x}) = f(V\vec{u}). \tag{$\star$}$$

You are now going to prove two theorems about the relationship between $g(\vec{u})$ and $f(\vec{x})$ assuming both ($\heartsuit$) and ($\star$). Although we generally discourage writing everything in coordinates, this is a case where it's helpful to do so, remembering the formulas for matrix-vector and matrix-matrix multiplication:

$$(A\vec{x})_i = \sum_j A_{ij}x_j \quad \text{and} \quad (AB)_{ij} = \sum_k A_{ik}B_{kj}$$

[f] That is $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$.

(1) (10 points) Use ($\star$) to prove that

$$\nabla g(\vec{u}) = \begin{bmatrix} \frac{\partial g}{\partial u_1}(\vec{u}) \\ \vdots \\ \frac{\partial g}{\partial u_n}(\vec{u}) \end{bmatrix} = V^T \begin{bmatrix} \frac{\partial f}{\partial x_1}(\vec{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\vec{x}) \end{bmatrix} = V^T \, \nabla f(\vec{x})$$

by proving that $\nabla g(\vec{u})_i = \frac{\partial}{\partial u_i} g(\vec{u}) = (V^T \nabla f(\vec{x}))_i$ for all i.

(2) (10 points) Now prove that

$$\mathcal{H}g(\vec{u}) = \left[\frac{\partial^2 g}{\partial u_i \partial u_j}(\vec{u})\right] = V^T \left[\frac{\partial^2 f}{\partial x_i \partial x_j}(\vec{x})\right] V = V^T \, \mathcal{H}f(\vec{x}) \, V$$

by proving that $(\mathcal{H}g(\vec{u}))_{ji} = \frac{\partial^2}{\partial u_j \partial u_i} g(\vec{u}) = (V^T \, \mathcal{H}f(\vec{x}) \, V)_{ji}$ for all $j, i \in 1, \ldots, n$ using the result of the last problem that $\frac{\partial}{\partial u_i} g(\vec{u}) = (V^T \nabla f(\vec{x}))_i$ for all $i \in 1, \ldots, n$.

3. (20 points) Recall that we gave in the notes:

Theorem (Spectral Theorem)**.** *If A is a real, symmetric $n \times n$ matrix then there exists a real, orthonormal basis $\vec{u}_1, \ldots, \vec{u}_n$ for $\mathbb{R}^n$ and real numbers $\lambda_1, \ldots, \lambda_n$ so that $A\vec{u}_i = \lambda_i \vec{u}_i$ for each $i \in 1, \ldots, n$.*

We claimed that this means "every symmetric matrix can be diagonalized". We're now going to explore exactly how this works in practice.

(1) (5 points) Suppose that A is an $n \times n$ symmetric matrix, and v is an orthogonal matrix whose columns $\vec{v}_1, \ldots, \vec{v}_n$ are orthonormal eigenvectors of A with eigenvalues $\lambda_1, \ldots, \lambda_n$. Prove that $V^T AV$ is a diagonal matrix where $(V^T AV)_{ii} = \lambda_i$ by proving that the vectors $\vec{e}_1, \ldots, \vec{e}_n$ are eigenvectors of $V^T AV$ with eigenvalues $\lambda_1, \ldots, \lambda_n$.

(2) (5 points) We also showed in the notes:
Proposition. *The eigenvalues of A are the roots of the polynomial* $\det(A - \lambda I)$.
Find the eigenvalues of the symmetric 2×2 matrix (in terms of the a_{ij})

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}.$$

and prove that the eigenvalues λ_1 and λ_2 are always real numbers if the a_{ij} are real.

(3) (5 points) In the case of 2×2 matrices, finding eigenvectors from eigenvalues is simple. Suppose we have

$$A - \lambda_1 I = \begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{12} & a_{22} - \lambda_1 \end{bmatrix}.$$

We know that $\det(A - \lambda_1 I) = 0$, so there is some $\vec{w}_1 \neq \vec{0}$ which solves the linear system $(A - \lambda_1 I)\vec{w}_1 = \vec{0}$. From this, we can construct

$$\vec{v}_1 = \frac{\vec{w}_1}{\|\vec{w}_1\|} \quad \text{and} \quad \vec{v}_2 = \vec{v}_1^{\perp}.$$

Use this procedure to find $\vec{v}_1$ and $\vec{v}_2$ (as functions of the λ_i and the a_{ij}).

(4) (5 points) Now we'll do an example. Use your work about to find eigenvalues λ_1 and λ_2 and corresponding orthonormal eigenvectors $\vec{v}_1$ and $\vec{v}_2$ for the symmetric matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

and construct the matrix V with columns $\vec{v}_1$ and $\vec{v}_2$. Then check directly that

$$V^T A V = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

4. (20 points) Assembling the pieces above, we now have a useful technique for simplifying quadratic polynomials in two variables. Much like completing the square, it's a way to change coordinates which can reduce any polynomial to a simpler form. Here is the plan:

- Given a polynomial $p(\vec{x}) = a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 + b_1x_1 + b_2x_2 + c$, form the matrix

$$A = 2\mathcal{H}p(\vec{x}) = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$$

  and find the orthogonal matrix V with columns given by the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of A.
- Make the change of variables $\vec{x} = V\vec{u}$.
- Observe that the new polynomial $p(\vec{u})$ has a diagonal Hessian. Since the Hessian is twice the coefficient matrix A, we should be able to write

$$p(\vec{u}) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + b_1' u_1 + b_2' u_2 + c$$

We are now going to do all of these steps in a specific example.

(1) (5 points) Form the coefficient matrix A for the quadratic polynomial

$$p(\vec{x}) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + 4x_1 + 3x_2 + 5$$

and write down the eigenvalues and eigenvectors of this matrix, using them to construct the matrix V.

(2) (5 points) Make the change of variables $\vec{x} = V\vec{u}$ and rewrite $p(\vec{x})$ as a new quadratic polynomial $p(\vec{u})$ in the new variables u_1 and u_2.

(3) (5 points) Check that $p(\vec{u})$ is in the form

$$p(\vec{u}) = \lambda_1 u_1^2 + \lambda_2 u_2^2 + b_1' u_1 + b_2' u_2 + c$$

and verify that $\vec{b}' = V^T \vec{b}$.

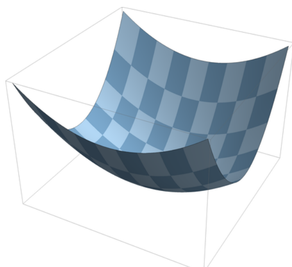
5. (10 points)

Definition. *A paraboloid with quadratic polynomial* $p(\vec{x})$ *is called an* elliptic *paraboloid if all the eigenvalues of* $\mathcal{H}p$ *are nonzero and have the same sign. If all of the eigenvalues of* $\mathcal{H}p$ *are equal, we call this an elliptic paraboloid* of revolution.

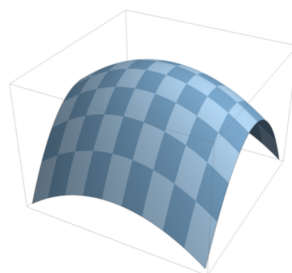
By choosing appropriate coordinates[h] for $\mathbb{R}^2$, every elliptic paraboloid $p(\vec{x})\colon\ \mathbb{R}^2 \to \mathbb{R}$ can be written in the form

$$p(\vec{x}) = \frac{\lambda_1}{2}x_1^2 + \frac{\lambda_2}{2}x_2^2 + c$$

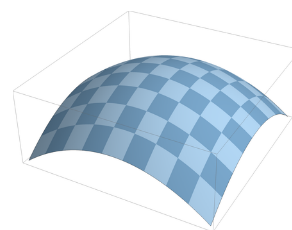
where λ_1, λ_2 are the eigenvalues of $\mathcal{H}p$ and $\lambda_1\lambda_2 > 0$. The surface curves up or down in both directions, as shown below.



$z = \frac{1}{5}x_1^2 + \frac{1}{2}x_2^2$

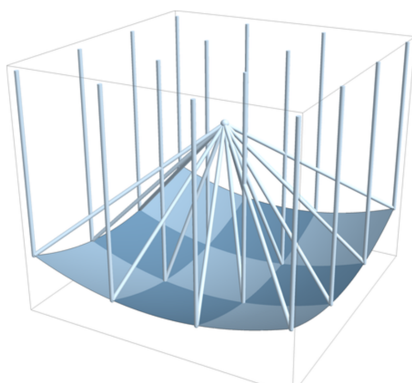


$z = -\frac{1}{5}x_1^2 - \frac{1}{2}x_2^2$



$z = -\frac{1}{5}x_1^2 - \frac{1}{5}x_2^2$

We are now going to prove that an elliptic paraboloid of revolution has the property that rays parallel to the z-axis reflected from the surface of the paraboloid meet at a single point on the z-axis, as shown below left.

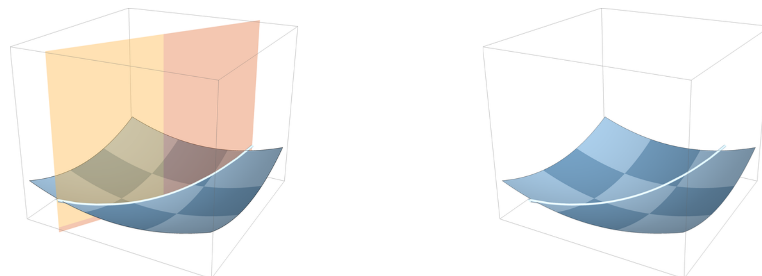


Paraboloids of this type are called "parabolic dishes" and are using to concentrate sound waves, radio waves, and light, particularly in solar energy applications, such as the solar furnace shown above right.

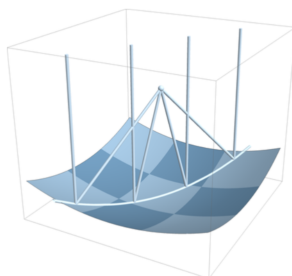
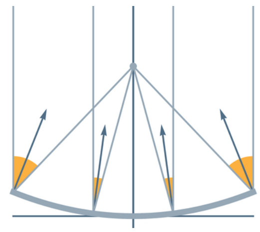
You should go about the proof in two steps.

[h]To be precise, the coordinate directions are the eigenvectors for $\mathcal{H}p$ and the origin of our "appropriate" coordinate system is chosen to be the (unique) point where $\nabla p = \vec{0}$.

(1) (5 points) Start by writing the function $p(\vec{x}) = \frac{\lambda_1}{2}x_1^2 + \frac{\lambda_2}{2}x_2^2 + c$ in polar coordinates (r, θ), using the fact that $\lambda_1 = \lambda_2$. Use the new equation to prove that the intersection of the paraboloid with any plane $\theta = \theta_0$ is a parabola coordinates, as shown below, and find the equation of that parabola.



(2) (15 points) You may assume that rays in the plane $\theta = \theta_0$ remain in that plane after reflection[i] as shown below left. This means that you can reduce the question to the planar problem shown below right.

You may also assume that the angles that the incoming and outgoing rays make with the normal vector to the parabola (shown in yellow above right) are equal.[j]

Prove that all the outgoing rays meet at a single point on the z-axis and give coordinates for this point in terms of the eigenvalues $\lambda_1 = \lambda_2$.

[i] This is true only because the elliptic paraboloid of revolution is rotationally symmetric.

[j] This is the law of reflection.

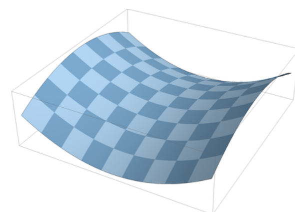
6. (10 points)

Definition. *A paraboloid* $p(\vec{x})$ *is called a* hyperbolic *paraboloid if the eigenvalues of* $\mathcal{H}p$ *are nonzero and do not all have the same sign.*

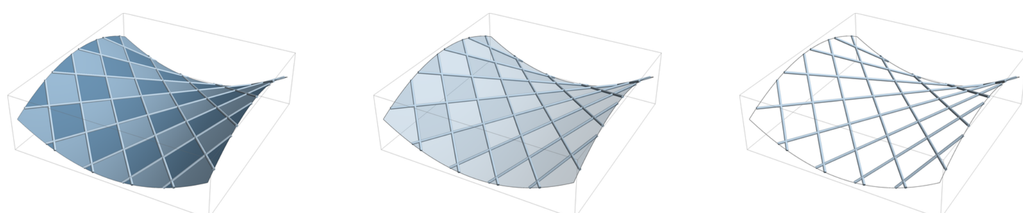
As before, in appropriate coordinates for $\mathbb{R}^2$, we can write $p(\vec{x})\colon\ \mathbb{R}^2 \to \mathbb{R}$ in the form

$$p(\vec{x}) = \frac{\lambda_1}{2}x_1^2 + \frac{\lambda_2}{2}x_2^2 + c$$

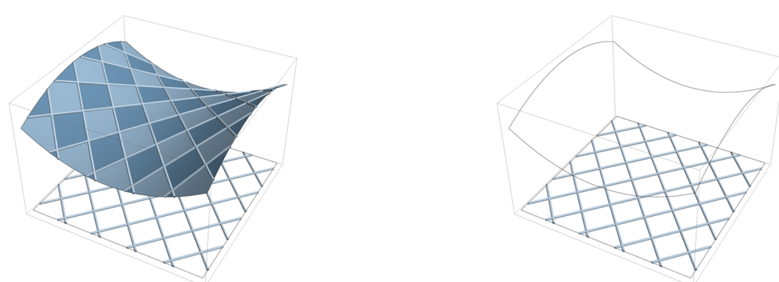
where λ_1, λ_2 are the eigenvalues of $\mathcal{H}p$ and $\lambda_1\lambda_2 < 0$. The surface curves up in one direction and down in the other, as shown below.



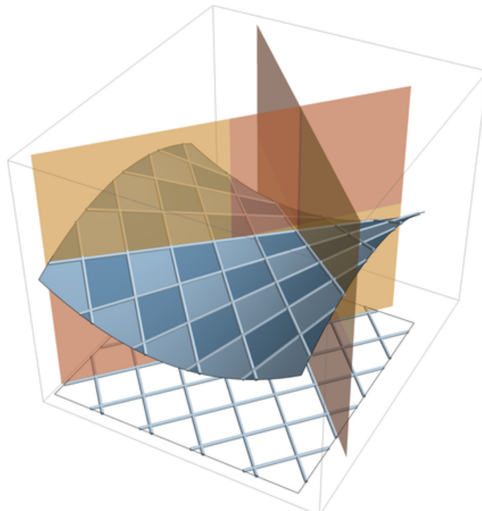
The hyperbolic paraboloid has the surprising property that it is composed of two families of straight lines, as shown below:



We are now going to prove this and find equations for the lines. Since the projection of a line in space to the x-y plane is a line in the plane, the projections of the lines making up the hyperboloid must form two families of parallel lines in the plane, as below.



(1) (7 points) Prove that there are exactly two slopes $\pm m$ for lines in the plane so that the intersection of the vertical plane through **any** line with slope $\pm m$ and the hyperboloid is a straight line, as shown below.



You must find the slopes $\pm m$ explicitly in terms of λ_1 and λ_2.

Hint: Remember that every line in the x-y plane can be written in the form $y = mx + b$. When is the space curve formed by the intersection of the hyperboloid with the vertical plane through $y = mx + b$ a straight line?

(2) (3 points) Now write the two equations for the parametrized lines $\vec{\alpha}(x)$ which pass above $(0, b)$, and simplify them as much as you can.

7. (20 points) (Extra Credit) You proved in question 5 that parallel rays of light striking a perfectly reflective and perfectly shaped paraboloid should converge at a single point on the axis of revolution.[1]

   Construct or obtain a concave dish which is as close to being an elliptic paraboloid of revolution you can. Line it with something reflective. Rays of sunlight arriving on Earth are almost parallel, your shape is (hopefully!) pretty close to being a true elliptic paraboloid of revolution, and the reflective material is close to being a perfect reflector. All of these things mean that light rays striking your dish may converge to *nearly* the same point on the axis.

   Test how well your construction collects sunlight by (for instance) measuring the temperature at the focus of the paraboloid and comparing this to the air temperature around the dish. If your dish focuses light effectively, the temperature will be higher at the focus. Can you collect enough sunlight to melt a chocolate bar? Boil water? Toast a marshmallow? Submit a written description of your construction process and photographs of you with your construction and the temperature difference you measure.

   Construction notes: It is often possible to obtain old satellite dishes (for example, from the Dish network) very cheaply. Check Craigslist or Facebook marketplace. Other construction options include cardboard, paper, and lining a shallow dish with modeling clay and shaping it with a template. Aluminum foil is a good reflective material.

   Important safety notes: This experiment can work well enough to product a lot of heat. Be very careful (you're legal adults; please act accordingly). Do NOT make a paraboloid much larger than a pie plate. (Large satellite dishes covered in a good reflective material can be quite dangerous.) Keep your hands and hair well away from the focus of your dish, and wear oven mitts. Measure temperatures with a long thermometer. Do not boil water in a sealed container. Toast marshmallows on a stick.

[1] Very reasonably, this is called the *focus* of the paraboloid.

8. (20 points) (Extra Credit) Construct your own hyperbolic paraboloid using two sets of straight rods, as in the video posted above this homework.[m] Submit photographs of you with your construction and a description of your method.

[m]Bamboo cooking skewers joined by hot glue or wires work well for this, as do 1/4 inch dowels from a home improvement center. Pencils can work, but require a little more care in assembly and make a very flat hyperbolic paraboloid– they are really too thick and short to be very convenient.