Math 4250 Minihomework: The Square-Wheeled Car

In this minihomework, we'll build up to an understanding of rolling curves and the square-wheeled car problem. This is something between a lecture and a homework assignment– basically, I would usually walk through the solution with you as a lecture, pausing to ask questions and (hopefully) get you to most of the work yourself. In this format, I've just written up the proof as a series of leading questions, hoping that you'll be able to fill in the gaps as you go.

1. (15 points) (Autonomous differential equations) Recall that a first order differential equation for an unknown function $u(t)$ is an equation in the form

$$u'(t) = F(u(t), t)$$

(where F is some function of $u'(t)$ and t) and that the equation is *autonomous* if the right hand side can be written only in terms of $u(t)$ as

$$u'(t) = F(u(t)) \tag{$\star$}$$

As you learned in MATH 2700, every autonomous first-order ODE may be solved as follows

$$\begin{aligned} u'(t) &= F(u(t)) \\ \frac{u'(t)}{F(u(t))} &= 1 \\ \int \frac{u'(t)}{F(u(t))}\,dt &= \int 1\,dt \\ \int \frac{1}{F(u)}\,du &= t + C \\ G(u) &= t + C \end{aligned}$$

where $G(u)$ is any antiderivative of $\frac{1}{F(u)}$. If we can find an inverse function $G^{-1}(u)$ for $G(u)$, we can solve explicitly for u as

$$u(t) = G^{-1}(t + C) \tag{$\spadesuit$}$$

This is the most general solution to equation ($\star$). If we know some value $u(t_0) = u_0$ we can plug it in to ($\spadesuit$) to solve for k.

(1) (5 points) Find the most general solution to $u'(t) = u(t)^2$. Be sure to explain every step and follow the outline above to show that you understand the proof.

(2) (5 points) Suppose $u(t_0) = u_0$. Combine this with 1.1 to write down the unique solution to the differential equation with these initial conditions in terms of t_0 and u_0.

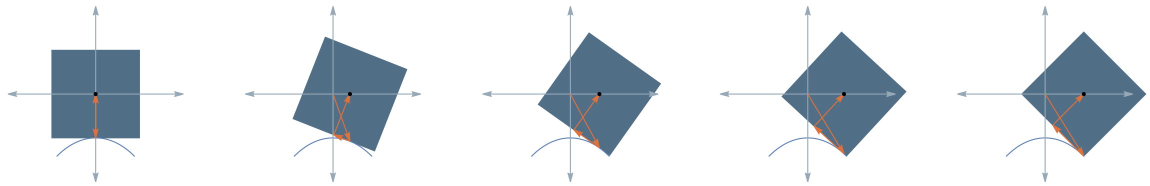
(3) (5 points) Suppose $t_0 = 0$ and $u_0 = 2$. Plug in to the results above to write down the particular solution with $u(0) = 2$.

2. (10 points) (The planar operator $\perp$) Suppose that $\vec{v} = (v_1, v_2)$ is vector in $\mathbb{R}^2$. We define the operator $\perp$ by $\vec{v}^{\perp} = (-v_2, v_1)$. The vector $\vec{v}^{\perp}$ is the counterclockwise rotation of $\vec{v}$ by $\pi/2$.

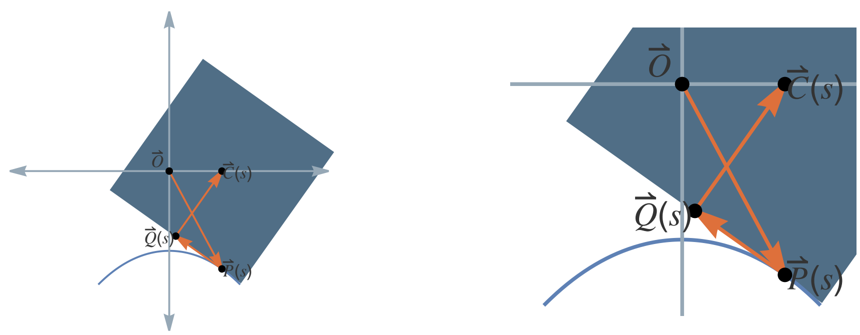
(1) (5 points) Show that $\left\langle \vec{v}, \vec{v}^{\perp} \right\rangle = 0$, and hence that $\vec{v}$ and $\vec{v}^{\perp}$ are orthogonal to one another.

(2) (5 points) Show that $\left|\left\langle \vec{v}, \vec{w}^{\perp} \right\rangle\right| = \|\vec{v}\| \|\vec{w}\| \sin\theta$, where $\theta \in [0, \pi]$ is the (unsigned) angle between $\vec{v}$ and $\vec{w}$. Hint: Write $\vec{v}$ and $\vec{w}$ in polar coordinates as $\vec{v} = \|\vec{v}\|(\cos\phi, \sin\phi)$ and $\vec{w} = \|\vec{w}\|(\cos\psi, \sin\psi)$.

3. (55 points) (The square-wheeled car) Consider the figure below, where a square with side-length 2 is rolling along a bumpy road. We want to think of the first arch of the road as the graph of a function $f(x)$ and solve for the function needed to make the wheel roll perfectly smoothly along the road.



We will parametrize the progress of the square by the arclength s it has rolled along the road since starting in the horizontal position with center $C(0) = O$. We will keep track of four points as the square rolls along the arch of the curve:



The point $\vec{O}$ is the origin. The point $\vec{P}(s)$ is the contact point between the square and the curve. The point $\vec{Q}(s)$ is the midpoint of the bottom side of the square, while the point $\vec{C}(s)$ is the center of the square.

For the square to roll smoothly, $\vec{C}(s)$ must stay on the x-axis:

$$\vec{C}(s) = (c_1(s), c_2(s)) = (c_1(s), 0)$$

Assume that $\vec{\alpha}(s)$ is an arclength parametrization of the road, so that $\vec{P}(s) = \vec{\alpha}(s) = (x(s), y(s))$, and assume that $\vec{\alpha}(0) = (0, -1)$. Further, assume that $\vec{Q}(s)$ is always the midpoint of that edge of the square. Since the square has rolled without slipping we know $\|\vec{Q}(s) - \vec{P}(s)\|$. Further, since the square cannot penetrate the curve, the curve determines the direction of the vector $\vec{Q}(s) - \vec{P}(s)$ as well.

Note: The parameter s is an arclength parameter for $\vec{\alpha(s)} = \vec{P}(s)$ (only). That is, $\vec{O}(s)$, $\vec{Q}(s)$, and $\vec{C}(s)$ certainly trace out parametrized curves in the plane as the square wheel turns. But we don't have any reason to believe that *those* curves are arclength parametrized.

(1) (5 points) Write down a formula for $\vec{P}(s) - \vec{O}(s)$ in terms of $x(s)$, $y(s)$, and s. (This is intended to be easy, so be careful that you're not overcomplicating things.) Explain why your formula is correct using at least one sentence.

(2) (5 points) Write down a formula for $\vec{Q}(s) - \vec{P}(s)$ in terms of $x(s)$, $y(s)$, and s. Explain using at least one sentence why your formula is correct.

(3) (5 points) Write down a formula for the vector from the center of the bottom side $\vec{Q}(s)$ to the center of the square $\vec{C}(s)$, that is, for $\vec{C}(s) - \vec{Q}(s)$, in terms of $x(s)$, $y(s)$, and s. Explain using at least one sentence why your formula is correct, being sure to point out where you used the hypothesis that the sidelength of the square was 2.

Hint: Remember the $\perp$ operator. Don't forget that the sides of the square have length 2.

(4) (5 points) Add the results of (a)-(c) to find a formula for $\vec{C}(s) - \vec{O}(s) = (c_1(s), c_2(s))$ in terms of $x(s)$, $y(s)$ and s. Since $c_2(s)$ (the height of the axle) is a constant function, we can compute the derivative $c_2'(s) = 0$ to get a relationship between s and $x''(s)/y''(s)$. Do so, including an explanation of your work as you do the computations.

(5) (5 points) Remember that $1 = \|\vec{\alpha}'(s)\|^2 = x'(s)^2 + y'(s)^2$. Differentiate both sides with respect to s and solve for $x''(s)/y''(s)$.

(6) (5 points) Combine the results of (d) and (e) and solve for the slope $y'(s)/x'(s)$ of the tangent line to $\vec{\alpha}(s)$.

(7) (5 points) Now there is some function f so that $y(s) = f(x(s))$. Differentiating both sides with respect to s), we know that

$$y'(s) = f'(x(s))x'(s)$$

Use this and the results of (f) to find a formula for $f'(x)$ in terms of s.

(8) (5 points) Now we reparametrize $\vec{\alpha}(s)$ as $\vec{\beta}(x) = (x, f(x))$. Now $s(x)$ is the arclength along $\vec{\beta}$ between 0 and x. Thus

$$s(x) = \int_0^x \|\vec{\beta}'(t)\| \, dt$$

and so

$$s'(x) = \|\vec{\beta}'(x)\|$$

Use the parametrization for $\beta(x)$ to write a formula for $s'(x)$ in terms of $f'(x)$.

(9) (10 points) Differentiate the results of (g) by x (on both sides) to get an expression for $f''(x)$ in terms of $s'(x)$, and use the results of (h) to write $f''(x) = F(f'(x))$ for some function F. Solve this autonomous differential equation for $f'(x)$ using the technique you reviewed in Problem 1 and the fact that you know $f'(0)$. You might have to look at the previous minihomework to do an integral ...

(10) (5 points) Integrate your formula for $f'(x)$ to (finally!) get $f(x)$. Use the fact that you know $f(0)$ to solve for the constant of integration. Celebrate your victory!

Hint: Don't forget that the square has sidelength 2 when you're computing $f(0)$.