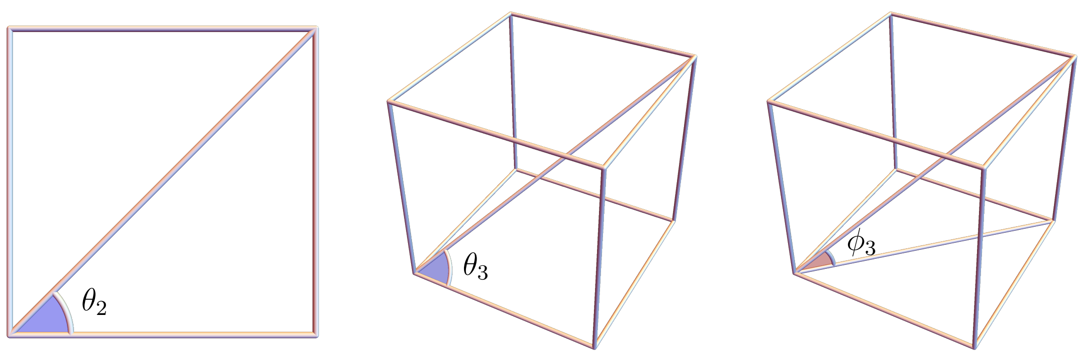
Math 4250/6250 Mini-Homework 1: Scalar and vector products

This minihomework accompanies the lecture notes on "Parametrized curves, scalar and vector products". A minihomework consists of a small number of homework problems which you should complete as soon as possible after class (while everything is still fresh in your mind). We'll sometimes start minihomeworks in class when we have time.

1. (8 points) The dot product is often used to compute angles in space via the formula

$$\langle \overrightarrow{v}, \overrightarrow{w} \rangle = \|\overrightarrow{v}\| \|\overrightarrow{w}\| \cos\theta.$$

We can define the n-dimensional hypercube to be the region with 2^n vertices $(\pm 1, \ldots, \pm 1)$. The 2-dimensional hypercube (the square), and the 3-dimensional hypercube (the cube) are pretty familiar to us.



We now define three vectors for a hypercube:

1. The edge vector $\overrightarrow{\mathrm{edge}}_n$ joins the corner $(-1, \ldots, -1)$ to the corner $(+1, -1, \ldots, -1)$.
2. The face diagonal vector $\overrightarrow{\mathrm{face}}_n$ joins $(-1, \ldots, -1)$ to $(+1, \ldots, +1, -1)$.
3. The main diagonal vector $\overrightarrow{\mathrm{diag}}_n$ joins $(-1, \ldots, -1)$ to $(+1, \ldots, +1)$.

The angle marked in blue, which we call θ_n, is the angle between $\overrightarrow{\mathrm{edge}}_n$ and $\overrightarrow{\mathrm{diag}}_n$, while the angle marked in pink, which we call ϕ_n, is the angle between $\overrightarrow{\mathrm{face}}_n$ and $\overrightarrow{\mathrm{diag}}_n$.

(1) (2 points) Find $\overrightarrow{\mathrm{edge}}_n$, $\overrightarrow{\mathrm{face}}_n$, and $\overrightarrow{\mathrm{diag}}_n$.

(2) (2 points) Use the dot product formula to find the cosine of θ_n in terms of n and then take $\lim_{n\to\infty} \cos\theta_n$. Use the fact that $\lim_{n\to\infty} \cos\theta_n = \cos(\lim_{n\to\infty} \theta_n)$ to compute $\lim_{n\to\infty} \theta_n$.

(3) (2 points) Use the dot product formula to find the cosine of ϕ_n in terms of n and then take $\lim_{n\to\infty} \cos\phi_n$. Use the fact that $\lim_{n\to\infty} \cos\phi_n = \cos(\lim_{n\to\infty} \phi_n)$ to compute $\lim_{n\to\infty} \phi_n$.

(4) (2 points) (Short answer) What do your values of $\lim_{n\to\infty} \theta_n$ and $\lim_{n\to\infty} \phi_n$ tell you about the geometry of the n-dimensional hypercube for large n? How does it differ from the geometry of the 2 or 3-dimensional cube?

2. (5 points) We claimed in class that if $\overrightarrow{\alpha}(t), \overrightarrow{\beta}(t) : \mathbb{R} \to \mathbb{R}^n$, then

$$\frac{d}{dt}\left\langle \overrightarrow{\alpha}(t), \overrightarrow{\beta}(t) \right\rangle = \left\langle \overrightarrow{\alpha}'(t), \overrightarrow{\beta(t)} \right\rangle + \left\langle \overrightarrow{\alpha}(t), \overrightarrow{\beta}'(t) \right\rangle.$$

Prove it, pointing out where you use the product rule $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

3. (5 points) We claimed in class that if $\overrightarrow{\alpha}(t), \overrightarrow{\beta}(t) : \mathbb{R} \to \mathbb{R}^3$, then

$$\frac{d}{dt}\overrightarrow{\alpha}(t) \times \overrightarrow{\beta}(t) = \overrightarrow{\alpha}'(t) \times \overrightarrow{\beta(t)} + \overrightarrow{\alpha}(t) \times \overrightarrow{\beta}'(t).$$

Prove it, pointing out where you use the product rule[1] $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$.

[1]The product rule for scalar functions is sometimes written $\frac{d}{dx}f(x)g(x) = f'(x)g(x) + g'(x)f(x)$. We now see why that's a bad idea:

$$\frac{d}{dt}\overrightarrow{\alpha}(t) \times \overrightarrow{\beta}(t) \neq \overrightarrow{\alpha}'(t) \times \overrightarrow{\beta(t)} + \overrightarrow{\beta}'(t) \times \overrightarrow{\alpha}(t).$$

4. (5 points) A very useful identity in dealing with cross products is the "bac-cab" identity:

$$\overrightarrow{a} \times \left(\overrightarrow{b} \times \overrightarrow{c} \right) = \overrightarrow{b} \left\langle \overrightarrow{a}, \overrightarrow{c} \right\rangle - \overrightarrow{c} \left\langle \overrightarrow{a}, \overrightarrow{b} \right\rangle .$$

Carefully prove "$\overrightarrow{b}$ and $\overrightarrow{c}$ are linearly dependent $\iff$ $\overrightarrow{b} \times \overrightarrow{c} = \overrightarrow{0}$" using the bac-cab identity. Be sure to use the definition of linear dependence:

Definition. *Two vectors $\overrightarrow{b}$ and $\overrightarrow{c}$ are linearly dependent $\iff$ there are coefficients λ and μ, not both zero, so that $\lambda \overrightarrow{b} + \mu \overrightarrow{c} = \overrightarrow{0}$.*

5. (5 points) (Challenge Problem) Recall that given any vectors $\overrightarrow{u}$ and $\overrightarrow{a}$, we can write $\overrightarrow{u}$ as a sum of components $\overrightarrow{u}_{\text{par}}$ and $\overrightarrow{u}_{\text{perp}}$ which are parallel and perpendicular to $\overrightarrow{a}$.

$$\overrightarrow{u} = \underbrace{\frac{\langle \overrightarrow{u}, \overrightarrow{a} \rangle}{\langle \overrightarrow{a}, \overrightarrow{a} \rangle} \overrightarrow{a}}_{\overrightarrow{u}_{\text{par}}} + \underbrace{\overrightarrow{u} - \frac{\langle \overrightarrow{u}, \overrightarrow{a} \rangle}{\langle \overrightarrow{a}, \overrightarrow{a} \rangle} \overrightarrow{a}}_{\overrightarrow{u}_{\text{perp}}}.$$

Definition. *If $\overrightarrow{v}$ is a vector in $\mathbb{R}^3$ and $\overrightarrow{a}$ is a unit vector in $\mathbb{R}^3$, then the* rotation of $\overrightarrow{v}$ around axis $\overrightarrow{a}$ by angle θ *is the unique vector $\overrightarrow{w}$ with*

1. $\|\overrightarrow{v}\| = \|\overrightarrow{w}\|$,
2. $\overrightarrow{v}_{par} = \overrightarrow{w}_{par}$,
3. *the angle between $\overrightarrow{v}_{perp}$ and $\overrightarrow{w}_{perp}$ is θ.*

where $\overrightarrow{v}_{par}$, $\overrightarrow{v}_{perp}$, $\overrightarrow{w}_{par}$ and $\overrightarrow{w}_{perp}$ are all computed with respect to the axis $\overrightarrow{a}$.

This definition is useful for proofs, but it doesn't give us any idea how to construct such a $\overrightarrow{w}$ given $\overrightarrow{v}$, $\overrightarrow{a}$ and θ, or even prove that any $\overrightarrow{w}$ with these properties exists.

Fix these problems by proving that the vector given by the formula

$$r(\overrightarrow{v}) = \overrightarrow{v} \cos\theta + (\overrightarrow{a} \times \overrightarrow{v}) \sin\theta + \overrightarrow{a} \langle \overrightarrow{a}, \overrightarrow{v} \rangle (1 - \cos\theta),$$

satisfies 1, 2, and 3, and hence is the rotation of $\overrightarrow{v}$ around the axis $\overrightarrow{a}$ by angle θ.