Math 4500/6500 Minihomework: Romberg Integration

This minihomework covers our notes on Romberg integration:

1. Simpson's rule. Simpson's rule is a different formula for numerical integration of $\int_a^b f(x)dx$ which is based on approximating $f(x)$ with a piecewise quadratic function. We will now derive Simpson's rule and relate it to Romberg integration:

   a. Suppose that $q(x)$ is a quadratic polynomial so that $q(-h) = f(-h)$, $q(0) = f(0)$ and $q(h) = f(h)$. Prove that
   $$\int_{-h}^{h} q(x)\,dx = \frac{h}{3}\left(f(-h) + 4f(0) + f(h)\right).$$

   b. Suppose that the interval $[a, b]$ is divided by $a = x_0, x_1, \ldots, x_{2n} = b$ into $2n$ intervals of equal width h. Suppose that $Q(x)$ is the piecewise quadratic function defined by $Q(x_i) = f(x_i)$ for all $i \in 0, \ldots, 2n$ and $Q(x)$ is quadratic on $[x_{2k}, x_{2k+2}]$ for all $k \in 0, \ldots, n$. Prove that
   $$\int_a^b Q(x)\,dx = \frac{h}{3}\left(f(a) + 4f(x_1) + 2f(x_2) + \cdots + 2f(x_{2n-2}) + 4f(x_{2n-1}) + f(x_{2n})\right).$$
   The approximation $\int_a^b Q(x)\,dx \sim \int_a^b f(x)\,dx$ is called "Simpson's rule".

   c. Prove that the entry $R(n, 1)$ in the Romberg table is equal to the result of Simpson's rule with 2^n intervals.

2. Referring to our formula for the error in Romberg integration (page 8 of the notes, also Theorem 5.2.1 in the book chapter from Dalquist and Björck), we will consider the error in estimating three different integrals with Romberg integration.
   $$\int_0^1 \frac{e^x}{2\sqrt{x}}\,dx, \quad \int_0^1 e^{(x^2)}\,dx \quad \text{and} \quad \int_0^1 \sqrt{x}\,dx.$$

   a. Find a bound (if one exists) on the $2k$-th derivative of each of the integrands above.

      Note: For the function $\sqrt{x}$, the right way to do this is to write down a general formula for the derivative. For the other two functions, try writing out the Taylor series of each function and using that to determine the derivatives.

   b. Predict which integrand will have the least truncation error
   $$\left|R(n, n-1) - \int_0^1 f(x)\,dx\right|$$
   and explain your reasoning in one paragraph or more.

   c. Use the Romberg integration code in the demonstration notebook to evaluate each of these three integrals (at various orders n). For which function does Romberg actually perform best? Does Romberg integration improve as you increase n? Explain.