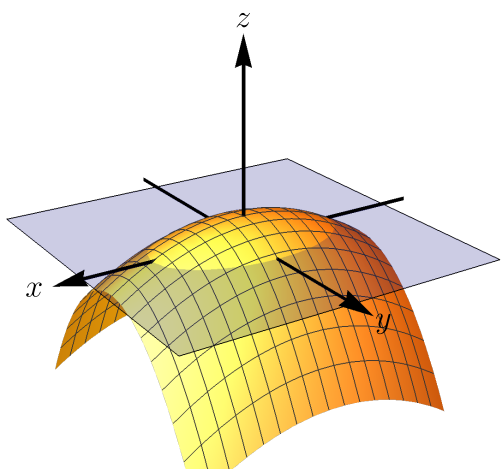
# Math 4250/6250 Mini-Homework

This minihomework accompanies the lecture notes on “A (lengthy) example”. However, it turns out that these notes are currently incomplete: you’ll have to do the reading in Shifrin to understand everything that we’re computing below.

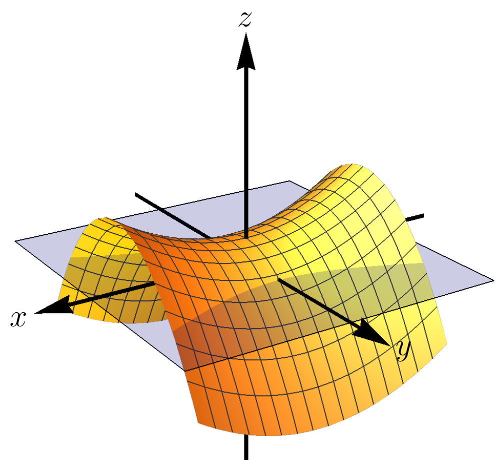
In this minihomework, we’ll do a lot of computations for an example which turns out to be, in a real sense, the general case:

$$\vec{x}(u, v) = (u, v, C + au^2 + bv^2)$$

For the duration of this homework, we’ll assume that $C$, $a$, and $b$ are real constants which may be positive, negative or zero. (We’ll see that each of these cases has consequences for the geometry of the surface.) To get oriented, we look at two examples of surfaces of this type:




$\vec{x}(u, v) = (u, v, \frac{1}{6} - \frac{1}{3}x^2 - y^2)$




$\vec{x}(u, v) = (u, v, \frac{1}{6} + \frac{1}{3}x^2 - y^2)$

1. Compute $\vec{x}_u(u, v)$, $\vec{x}_v(u, v)$ and $\vec{n} = \frac{\vec{x}_u \times \vec{x}_v}{\|\vec{x}_u \times \vec{x}_v\|}$.

2. Compute $E = \langle \vec{x}_u, \vec{x}_u \rangle_{\mathbb{R}^3}$, $F = \langle \vec{x}_u, \vec{x}_v \rangle_{\mathbb{R}^3}$, and $G = \langle \vec{x}_v, \vec{x}_v \rangle_{\mathbb{R}^3}$.

3. Compute $\vec{x}_{uu}(u, v)$, $\vec{x}_{uv}(u, v)$ and $\vec{x}_{vv}(u, v)$.

4. Compute $\ell = \langle \vec{n}, \vec{x}_{uu} \rangle_{\mathbb{R}^3}$, $m = \langle \vec{n}, \vec{x}_{uv} \rangle_{\mathbb{R}^3}$, $n = \langle \vec{n}, \vec{x}_{vv} \rangle_{\mathbb{R}^3}$.

5. Compute the symmetric matrices corresponding to the first and second fundamental form

$$\mathrm{I}_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix} \qquad \mathrm{II}_p = \begin{pmatrix} \ell & m \\ m & n \end{pmatrix}.$$

6. Compute the matrix[1] corresponding to the shape operator

$$S_p = \mathrm{I}_p^{-1}\, \mathrm{II}_p\,.$$

[1] As we saw in the last minihomework, even though the shape operator as a linear map is symmetric with respect to $\mathrm{I}_p$, this matrix won’t be symmetric unless $\mathrm{I}_p$ is a diagonal matrix (that is, $F = 0$).

7. Compute the eigenvalues $k_1$ and $k_2$ and eigenvectors $\vec{v}_1$ and $\vec{v}_2$ of the shape operator $S_p$. The eigenvalues $k_1$ and $k_2$ are the *principal curvatures*, while the eigenvectors $\vec{v}_1$ and $\vec{v}_2$ are the *principal directions*. (Follow the method in the notes to review how to find the eigenvalues and of a 2x2 matrix.)

8. Compute the Gauss curvature $K = k_1 k_2$ and Mean curvature $H = \frac{1}{2}(k_1 + k_2)$ of the surface.

9. How does the sign of the Gauss curvature and Mean curvature depend on the coefficients $C$, $a$, and $b$ which determine our surface? Which values of $C, a, b$ give us $K = 0$? Which values give us $H = 0$?

10. Compute the Gauss and Mean curvature of the two example surfaces for which we provide pictures at the start of the homework:

$$\vec{x}(u,v) = \left(u, v, \frac{1}{2} - \frac{1}{3}x^2 - y^2\right) \quad \text{and} \quad \vec{x}(u,v) = \left(u, v, \frac{1}{2} + \frac{1}{3}x^2 - y^2\right).$$