Math/Csci 4690/6690 : Linear Algebra of Eigenvalues and Eigenvectors

In this minihomework, we recall some facts about eigenvectors and eigenvalues that you (hopefully) covered in your linear algebra class.

1. (10 points) Suppose that M is a matrix, and let u and v be vectors so that

$$Mu = \lambda u \quad \text{and} \quad Mv = \mu v.$$

Prove that if M is symmetric and $\mu \neq \lambda$ then $\langle u, v \rangle = 0$.

Note: Make sure you use the fact that M is symmetric– the statement is false otherwise!

2. (20 points)

Definition. *An* $n \times n$ *matrix* Q *is* orthogonal *if* $Q^T = Q^{-1}$.

We note that this means that $QQ^T = Q^TQ = I_n$.

It is a fact that an orthogonal 2×2 matrix is a rotation or reflection, and an orthogonal 3×3 matrix is a rotation around some axis, possibly composed with a reflection in the plane normal to the axis of rotation.

For these reasons, we think of $n \times n$ orthogonal matrices as generalized "rotations", even though they may not have a single axis and angle[1].

(1) (10 points) Prove that if Q is orthogonal, then $\langle u, v\rangle = \langle Qu, Qv\rangle$. In particular, we have $\langle u, v\rangle = 0 \iff \langle Qu, Qv\rangle = 0$. This is why we call these matrices "orthogonal": they carry pairs of orthogonal vectors to pairs of orthogonal vectors.

[1] For example, an orthogonal matrix could act on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ by rotating the two copies of $\mathbb{R}^2$ by two different angles.

(2) (10 points) Show that if $Mv = \lambda v$ and Q is any orthogonal matrix then if $Mv = \lambda v$, we have $(QMQ^T)(Qv) = \lambda(Qv)$. That is, an orthogonal transformation of a matrix just rotates the eigenvectors; it doesn't change the eigenvalues.

3. (15 points) A permutation of a vector $v = (v_1, \ldots, v_n)$ is a rearrangement of its coordinates. For example (v_2, v_1, v_3) is a permutation of (v_1, v_2, v_3), as is (v_3, v_2, v_1). We can represent a permutation by a bijective function

$$\pi\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$$

We can write the action of a permutation on a vector as a matrix:

$$(\Pi)_{ij} = \begin{cases} 1, & \text{if } \pi(i) = j, \\ 0, & \text{otherwise} \end{cases}$$

For instance, the permutation $\pi(1, 2, 3) = (2, 1, 3)$ is encoded by the matrix

$$\Pi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as we can see by taking the product

$$\Pi v = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 \\ v_1 \\ v_3 \end{pmatrix}$$

(1) (10 points) Prove that permutation matrices are orthogonal matrices.

(2) (5 points) Use the result above and the result of problem 1 to show that if Π is a permutation matrix and $Mv = \lambda v$, we have

$$(\Pi M \Pi^T)(\Pi v) = \lambda(\Pi v)$$

That is, permuting the coordinates of a matrix just permutes the coordinates of the eigenvectors; it doesn't change the eigenvalues.

4. (15 points)

Definition. *Two* $n \times n$ *matrices* A *and* B *are* similar *if there is an invertible matrix* X *so that* $X^{-1}AX = B$.

Prove that if A and B are similar, then A and B have the same eigenvalues.

5. (15 points) (The Spectral Decomposition Theorem) Suppose that M is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and that $v_1, \ldots, v_n$ are a set of orthonormal column eigenvectors. Let V be the (orthogonal) matrix whose i-th column is v_i. Prove

$$V^T M V = \Lambda, \quad \text{where } \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n),$$

the diagonal $n \times n$ matrix with $\lambda_1, \ldots, \lambda_n$ on the diagonal[2].

[2]Note that this theorem implies that

$$M = V \Lambda V^T = \sum_i \lambda_i v_i v_i^T$$

where the $n \times n$ matrix $v_i v_i^T$ is the "outer product" of the column vector v_i with itself. (The "inner product" or dot product is the 1×1 matrix $v_i^T v_i$.) This description of M will often be helpful when M is the diffusion operator or graph Laplacian!

6. (15 points)

Definition. *The trace of an $n \times n$ matrix M is given by the sum of diagonal entries:*

$$\operatorname{tr} M := \sum M_{ii}$$

It is a helpful fact that for any pair of matrices,

$$\operatorname{tr} AB = \operatorname{tr} BA.$$

Use this fact and the previous exercise to show that if M is a symmetric $n \times n$ matrix and $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of M, then

$$\operatorname{tr} M = \lambda_1 + \cdots + \lambda_n.$$