Math 4250 Minihomework: The Frenet Frame without an Arclength Parametrization

In the last set of notes (and homework), we learned about the Frenet frame $T(s)$, $N(s)$, $B(s)$, which is an orthonormal frame for $\mathbb{R}^3$ defined at each point $\vec{\alpha}(s)$ of an arclength parametrized curve. The Frenet frame determines two scalar functions, curvature and torsion, which tell us about the shape of the curve. The most important fact about these is:

Proposition (The Frenet Equations)**.** *The curvature $\kappa(s)$ and the torsion $\tau(s)$ are defined by*

$$\begin{aligned} T'(s) &= && +\kappa(s)N(s) && \\ N'(s) &= -\kappa(s)T(s) && && +\tau(s)B(s) \\ B'(s) &= && -\tau(s)N(s) && \end{aligned}$$

In this set of notes, we learned several new ideas about the Frenet frame. Let's summarize:

Definition. [1] *The Frenet frame $T(t)$, $N(t)$ and $B(t)$ at $\vec{\alpha}(t)$ is the orthonormal basis for $\mathbb{R}^3$ obtained by Gram-Schmidt orthogonalization of the vectors $\vec{\alpha}'(t)$, $\vec{\alpha}''(t)$, and $\vec{\alpha}'(t) \times \vec{\alpha}''(t)$, in this order.*[2] *The Frenet frame is defined when all three vectors are defined and nonzero.*

Proposition. *The Frenet frame at $\vec{\alpha}(t)$ can be written as*

$$\begin{aligned} T(t) &= \frac{1}{\|\vec{\alpha}'\|}\vec{\alpha}' \\ N(t) &= -\frac{\langle \vec{\alpha}', \vec{\alpha}'' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\| \|\vec{\alpha}'\|}\vec{\alpha}' + \frac{\|\vec{\alpha}'\|}{\|\vec{\alpha}' \times \vec{\alpha}''\|}\vec{\alpha}'' \\ B(t) &= \frac{1}{\|\vec{\alpha}' \times \vec{\alpha}''\|}\vec{\alpha}' \times \vec{\alpha}''. \end{aligned}$$

where all the derivatives on the right hand sides are derivatives with respect to t.

Proposition. *If we are given any function $\vec{v}(t)\colon \mathbb{R} \to \mathbb{R}^n$ along a regular curve $\vec{\alpha}(t)$, then*

$$\frac{d}{ds}\vec{v}(t(s)) = \vec{v}'(t) \cdot \frac{1}{\|\vec{\alpha}'(t)\|}.$$

Here the derivatives on the right hand side are derivatives with respect to t. Notice that this theorem also holds for "vector" functions $v(t)\colon \mathbb{R} \to \mathbb{R}$.

Proposition. *The curvature and torsion are given in terms of t derivatives by*

$$\kappa(t) = \frac{\|\vec{\alpha}' \times \vec{\alpha}''\|}{\|\vec{\alpha}'\|^3} \quad \textit{and} \quad \tau(t) = \frac{\langle \vec{\alpha}', \vec{\alpha}'' \times \vec{\alpha}''' \rangle}{\|\vec{\alpha}' \times \vec{\alpha}''\|^2}$$

[1] It's basically a matter of taste whether this is the definition of the Frenet frame or a theorem about the Frenet frame.

[2] One interesting feature of Gram-Schmidt is that it depends on the order in which you present the vectors– for instance, the first vector is always part of the orthonormal basis output from Gram-Schmidt.

1. (20 points) (Signed curvature for non-arclength parametrized plane curves) Recall that in the last homework, we defined tangent and normal vectors for a plane curve $\vec{\alpha} : \mathbb{R} \to \mathbb{R}^2$ with a unit-speed parametrization

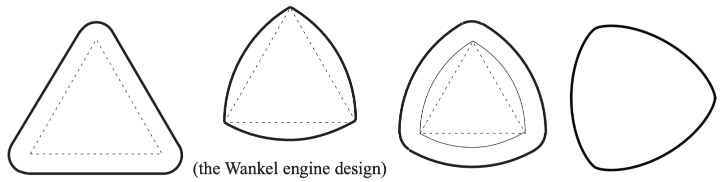
$$T(s) = \vec{\alpha}'(s), \quad \text{and} \quad N(s) = T(s)^{\perp}.$$

We also defined the signed curvature by

$$\kappa_{\pm}(s) = \langle T'(s), N(s) \rangle$$

Use the relation between s derivatives and t derivatives on the previous page to find an expression for $\kappa_{\pm}(t)$ when the curve $\vec{\alpha}(t)$ is not parametrized by unit speed.

2. (20 points) (Constant Breadth) A closed planar curve $\vec{\alpha}(s)$ is said to have constant breadth if the distance between parallel tangent lines of $\vec{\alpha}(s)$ is always μ. A circle is an example of such a curve, but it's not the only one:



We'll assume that $\vec{\alpha}(t) : [0, L] \to \mathbb{R}^2$ is an arclength parametrization of a curve of constant breadth $\vec{\alpha}(t)$ with $\vec{\alpha}(0) = \vec{\alpha}(L)$ (and all derivatives of $\vec{\alpha}(s)$ equal at 0 and L as well. We'll also assume that the signed curvature $\kappa(s) > 0$.

(1) (10 points) Let's call two points with parallel tangent lines *opposite* points. Suppose that $\vec{\beta}(t)$ is the opposite point to $\vec{\alpha}(t)$. We know that $\|\vec{\alpha}'(t)\| = 1$ because t is an arclength parameter for $\vec{\alpha}$.[3] Since the tangent and normal vectors $T(t)$, $N(t) = T(t)^{\perp}$ at $\vec{\alpha}(t)$ are a basis for the plane, there are some coefficients $c_1(t)$ and $c_2(t)$ so that

$$\vec{\beta}(t) - \vec{\alpha}(t) = c_1(t)T(t) + c_2(t)N(t).$$

Prove that $c_2(t) = \mu$ and then prove that $c_1(t) = 0$. Conclude that the chord joining opposite points is normal to the curve at both ends.

[3] However, there is no reason for t to also be an arclength parameter for the opposite points $\vec{\beta}(t)$, although this is true in the special case where α is the circle.

(2) (10 points) Let $\kappa_{\pm}^{\alpha}(t)$ be the signed curvature of $\vec{\alpha}$ at t, and $\kappa_{\pm}^{\beta}(t)$ be the signed curvature of $\vec{\beta}$ at t. Prove that the signed curvatures

$$\frac{1}{\kappa_{\pm}^{\alpha}(t)} + \frac{1}{\kappa_{\pm}^{\beta}(t)} = \mu$$

Hint: Let $\vec{T}_{\beta}(t)$ and $\vec{N}_{\beta}(t)$ denote the tangent and normal vectors to $\vec{\beta}(t)$. How are they related to $\vec{T}(t)$ and $\vec{N}(t)$? Can you use the results of 1 to compute $\kappa_{\pm}^{\beta}(t)$?

3. (10 points) (Torsion for a curve without an arclength parametrization) Suppose that $\vec{\alpha}(t)$ is a parametrization of a space curve $\vec{\alpha}$ with $\vec{\alpha}'(t) \neq \vec{0}$ and $\vec{\alpha}''(t) \neq 0$. Use the equations for $B(t)$ and $N(t)$ and the equation

$$\tau(t) = -\left\langle \frac{d}{ds} B(t), N(t) \right\rangle$$

to show that the torsion of $\vec{\alpha}(t)$ is given by:

$$\tau(t) = \frac{\langle \vec{\alpha}'(t), \vec{\alpha}''(t) \times \vec{\alpha}'''(t) \rangle}{\|\vec{\alpha}'(t) \times \vec{\alpha}''(t)\|^2}$$

Hint: You might want to review the properties of the *triple product* $\left\langle \vec{a}, \vec{b} \times \vec{c} \right\rangle$ from the very first set of notes before you get started.