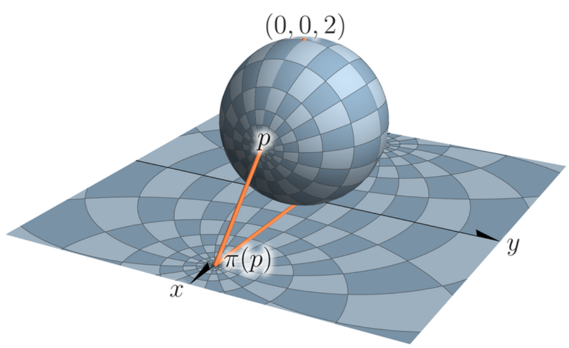
Math 4250 Minihomework: The First Fundamental Form

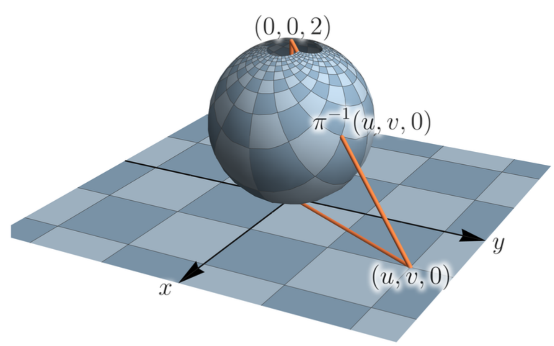
Consider the sphere $x^2 + y^2 + (z-1)^2 = 1$ centered at $(0,0,1)$ with radius 1, which we will call $S^2(0,0,1)$.

Definition. *We define the map* $\pi\colon\ S^2(0,0,1) \to \mathbb{R}^2$ *by taking* $\pi(\vec{p})$ *to be the unique point on the* x-y *plane which lies on the line through* $\vec{p}$ *and the north pole* $(0,0,2)$ *of* $S^2(0,0,1)$. *This map is called* stereographic projection.

Definition. *We define the map* $\pi^{-1}\colon\ \mathbb{R}^2 \to S^2(0,0,1)$ *by taking* $\pi^{-1}(u,v)$ *to be the unique point on* $S^2(0,0,1)$ *which lies on the interior of the line segment joining* $(u,v,0)$ *and* $(0,0,2)$. *This map is called* inverse stereographic projection.

The pictures below show stereographic projection from a point p on the sphere to a point $\pi(p)$ on the plane (left) and inverse stereographic projection from a point $(u,v,0)$ to a point $\pi^{-1}(u,v,0)$ (right). In each picture, the mesh lines on the sphere are mapped to the mesh lines on the plane by stereographic projection, giving us a sense of the entire mapping.





Straight lines in the plane map to circles on the sphere which pass through the north pole and other circles on the sphere map to circles in the plane. Further, you can see from the picture that right angles between the mesh boundaries on the plane map to right angles between mesh boundaries on the sphere.

The mapping on the left hand side establishes *bipolar coordinates* on the plane, which I encourage you to look up, because they are very cool. For a nice example of the mapping on the right hand side, watch this very short video by my friend Henry Segerman (click here), which is also linked on the course page.

1. (30 points) We will now prove a few facts about stereographic projection.

   (1) (10 points) Prove that the inverse stereographic projection map $\pi^{-1}\colon \mathbb{R}^2 \to S^2 - (0,0,2)$ is given explicitly in coordinates by the formula

$$\pi^{-1}(u,v,0) = \left(\frac{4u}{u^2+v^2+4}, \frac{4v}{u^2+v^2+4}, \frac{2(u^2+v^2)}{u^2+v^2+4}\right)$$

   Hint: The line from $\vec{a}$ to $\vec{b}$ is parametrized by $\vec{\alpha}(t) = (1-t)\vec{a} + t\vec{b}$.

(2) (10 points) Show that the map $X\colon \mathbb{R}^2 \to S^2 - (0,0,2)$ given by $X(u,v) = \pi^{-1}(u,v,0)$ is a regular parametrization of (almost all of) the sphere.

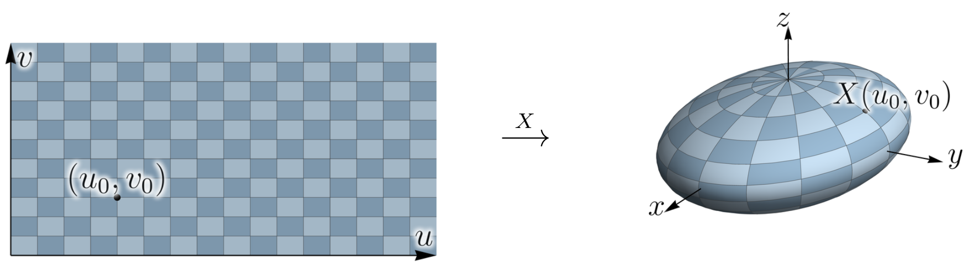
(3) (10 points) Find the first fundamental form I_p of this parametrization and show that it is a scalar multiple of the 2×2 identity matrix.

2. (40 points) Find the entries $E = \langle X_u, X_u \rangle$, $F = \langle X_u, X_v \rangle$, and $G = \langle X_v, X_v \rangle$ for the first fundamental form I_p of the parametrized surfaces below. Because the expressions for these tend to be complicated, you don't have to write I_p in the matrix form $I_p = \left[\begin{smallmatrix} E & F \\ F & G \end{smallmatrix}\right]$, though you should remember that I_p is the quadratic form determined by this matrix.

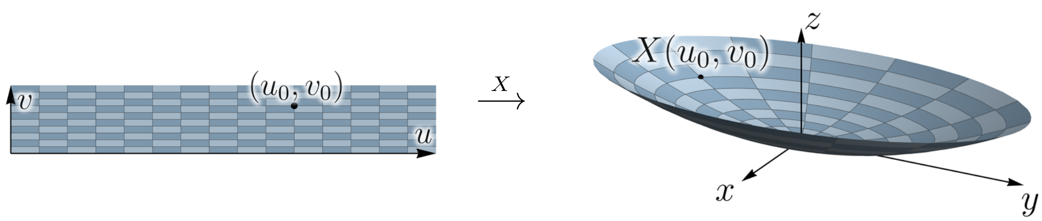
   You may assume that a, b, and c are constants.

   (1) (10 points) The ellipsoid.

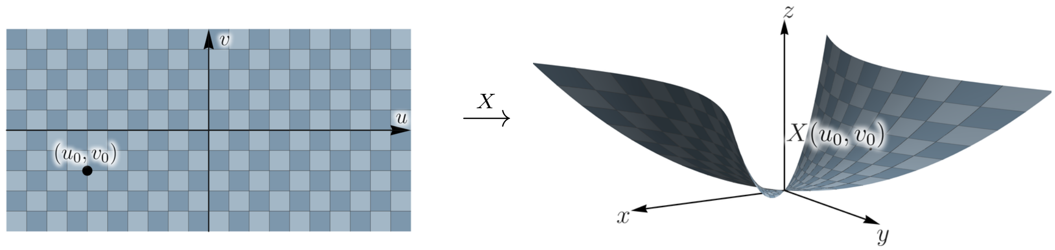
$$X(u, v) = (a \sin v \cos u, b \sin v \sin u, c \cos u), \quad (u, v) \in (0, 2\pi) \times (0, \pi).$$



(2) (10 points) The elliptic paraboloid. $X(u, v) = (av \cos u, bv \sin u, v^2)$.



(3) (10 points) The hyperbolic paraboloid. $X(u, v) = (au \cosh v, bu \sinh v, u^2)$.



(4) (10 points) The hyperboloid of two sheets. $X(u,v) = (a \sinh v \cos u, b \sinh v \sin u, \pm c \cosh v)$.