Math 6250 Graduate Material and Minihomework
Diagonalizing a quadratic

In the regular minihomework for this section, we stated (without proof!) that the function

$$\vec{x}(u,v) = (u, v, C + au^2 + bv^2)$$

was "in a real sense, the general case". For the benefit of the graduate students, we'll now explain why this is and work through some examples as minihomework.

1. Quadratic functions

The first thing we'll consider is the theory of a general quadratic function:

$$f(x,y) = \alpha x^2 + \beta y^2 + \gamma xy. \tag{1}$$

We want to show that by an orthogonal change of basis for the x-y plane, we can always change a function in the form (1) to an equation in the form

$$g(z,w) = az^2 + bw^2. \tag{2}$$

We'll do this in a couple of stages. We start by defining a matrix

$$Q_f = \begin{pmatrix} \alpha & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & \beta \end{pmatrix}. \tag{3}$$

1.1. **Problems.**

1. Prove that

$$f(x,y) = \begin{pmatrix} x & y \end{pmatrix} Q_f \begin{pmatrix} x \\ y \end{pmatrix}$$

2. Suppose that A is a 2×2 orthogonal matrix so that $\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} z \\ w \end{pmatrix}$. Show that if we rewrite $f(x,y)$ as a function $g(z,w)$ then the new function is given by

$$g(z,w) = \begin{pmatrix} z & w \end{pmatrix} A^T Q_f A \begin{pmatrix} z \\ w \end{pmatrix}$$

3. We know that a symmetric matrix has real eigenvalues, so there is an orthogonal matrix A with columns A_1 and A_2 which are eigenvectors of Q_f with eigenvalues λ_1 and λ_2. Prove that if we use such a matrix A in the last problem, then

$$g(z,w) = \lambda_1 z^2 + \lambda_2 w^2.$$

2. TAYLOR'S THEOREM

Suppose that we have a regular parametrized surface M and a point $p \in M$. We can assume (by rotating $\mathbb{R}^3$ if needed) that we have a parametrization

$$\vec{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

of M so that $\vec{x}(0,0) = \vec{p}$, the tangent plane $T_{\vec{0}}$ is the x-y plane, and $\vec{n}(0,0) = \hat{z} = (0,0,1)$.

We are going to prove that we can write M locally as the graph surface

$$\vec{y}(u,v) = (u, v, g(u,v) + \epsilon(u,v)) \quad \text{where} \quad g(u,v) = au^2 + bv^2 \quad \text{and} \quad \lim_{u,v\to 0} \frac{\epsilon(u,v)}{u^2+v^2} = 0. \tag{4}$$

We will assume that $\vec{x}(u,v)$ is C^2 so that we can use Taylor's theorem:

Theorem 1. *If a function $f:\mathbb{R}^2 \to \mathbb{R}$ is C^2, then we may write*

$$f(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \\ + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right) + \epsilon(x,y).$$

where $\lim_{x,y\to 0} \frac{\epsilon(x,y)}{x^2+y^2} = 0$.

and the inverse function theorem

Theorem 2. [1] *Let $f:\Omega \to \mathbb{R}^n$ where $\Omega \subset \mathbb{R}^n$ is an open set be a C^k function (for $k \geq 1$) and p a point in Ω where the Jacobian matrix $D_p f$ is invertible. Then there is an open set X containing p and an open set Y containing $f(p)$ and a C^k function $g:Y \to X$ so that $f(g(y)) = y$ for all $y \in Y$ and $g(f(x)) = x$ for all $x \in X$.*

2.1. Problems.

1. Let's restrict our attention to the first two coordinate functions $x_1(u,v)$ and $x_2(u,v)$ and use them to define a function $f(u,v) = (x_1(u,v), x_2(u,v))$ from the u-v plane to the x-y plane.

   Apply the inverse function theorem to show that f has a C^2 inverse function f^{-1} which allows us to define
   $$\vec{y}(u,v) = (u, v, x_3(f^{-1}(u,v)))$$
   and show that $\vec{y}$ and $\vec{x}$ parametrize the same surface M.

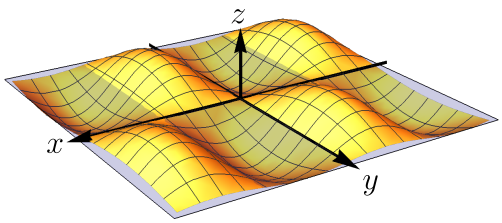
2. Prove that Taylor's theorem applies to $x_3(f^{-1}(u,v))$, and use it to prove that
   $$x_3(f^{-1}(u,v)) = g(u,v) + \epsilon(u,v)$$
   where $g(u,v) = \alpha\, u^2 + \beta\, v^2 + \gamma\, uv$ and $\lim_{u,v\to 0} \frac{\epsilon(u,v)}{u^2+v^2} = 0$.

3. Use the results of the first problem set to show that there is a rotation of $\mathbb{R}^3$ which puts $x_3(f^{-1}(u,v))$ in the form of (4). (Be a little careful here: you have to show that the *new* ϵ function still obeys $\lim_{x,y\to 0} \frac{\epsilon(u,v)}{u^2+y^2}$.)

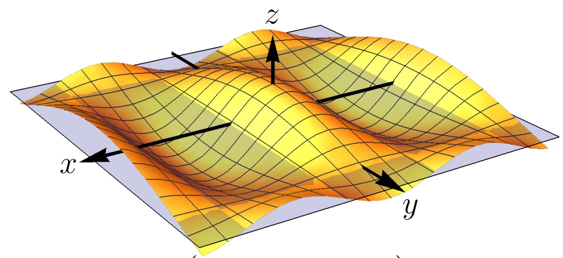
[1] For instance, Theorem 6 in the supplemental reading by Oliveira.

3. EXAMPLES

We can now look at some new examples:



$\vec{x}(u,v) = \left(u, v, -\sin x \sin \frac{y}{2}\right)$



$\vec{x}(u,v) = \left(u, v, \cos x \cos \frac{y}{3}\right)$

1. Explicitly reparametrize each of these surfaces around $\vec{0}$ in the form
$$\vec{y}(u,v) = (u, v, au^2 + bv^2 + \epsilon(u,v)).$$
Note: The left-hand surface will be slightly more challenging that the right-hand one.

2. Prove that the $\epsilon(u,v)$ functions that you give above have $\lim_{u,v\to 0} \frac{\epsilon(u,v)}{u^2+v^2}$.

3. Graph the original surfaces and the approximating surfaces given by
$$\vec{y}(u,v) = (u, v, au^2 + bv^2)$$
in a neighborhood of $\vec{0}$ using Mathematica (or your favorite graphing program, such as Desmos). Make sure that the quadratic approximations are very close to the originals!