Math 4500/6500 Minihomework: Derivative Formulae from Polynomial Interpolation

This homework assignment covers our notes on "Estimating Derivatives by Polynomial Fitting". If you find the algebra lengthy or irritating (which is pretty likely), you are encouraged to use *Mathematica* to do it.

We have covered in the notes the problem of finding a derivative formula for equally spaced data. This is definitely algebraically convenient, and it would be hard to imagine exactly how to deal with arbitrarily spaced data by manipulating Taylor series. However, sometimes your data just isn't equally spaced and you can't resample. For example, your data is from sensor readings in a robot which happen "when the CPU has time", or your data is from historical records such as temperature measurements which weren't taken at the same intervals.

This case is where finding derivatives by polynomial interpolation really does well. It turns out that for unequally spaced points $x_0 < x_1 < x_2$ where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$ (instead of h) we have

$$f''(x) \sim \frac{2}{h^2}\left[\frac{f(x_0)}{1+\alpha} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(\alpha+1)}\right]$$

The minihomework is to prove this by each of the following methods:

1. Approximate $f(x)$ by the Newton form of the interpolating polynomial of degree 2 which interpolates f at x_0, x_1, and x_2 and show that the expression above is the second derivative of that polynomial.

2. Suppose that the approximation formula was in the general form

   $$f''(x) \sim Af(x_0) + Bf(x_1) + Cf(x_2)$$

   and solve for the undetermined coefficients by making this formula exact for the three polynomials 1, $x - x_1$, and $(x - x_1)^2$ (and concluding that the formula is therefore exact for all polynomials of degree ≤ 2 by linearity).

3. (Challenge) Use Mathematica's `InterpolatingPolynomial` command to get a (symbolic) interpolating polynomial for the general data $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$, and then make the substitutions $x_1 - x_0 \to h$ and $x_2 - x_1 \to \alpha h$ and simplify to get the form above. (This is probably the hardest of the three methods because it requires a fair amount of fooling around with replacement rules and the like to get Mathematica to express the results in terms of α.)