Math 4250 Minihomework: Curve Theory concluded

In this minihomework, we are going to wrap up our study of the differential geometry of curves by learning a few useful techniques for studying plane curve (and convex curves in particular).

1. (10 points) Let's review a few facts about lines.

   Definition. *A line ℓ in $\mathbb{R}^2$ may be expressed in* parametric form *as the set of points*

   $$\vec{\alpha}(s) = \vec{m}s + \vec{b},$$

   where $\vec{m}$ is the direction of the line, $\vec{b}$ is a point on the line, and s is any real number. We may also express ℓ in implicit form *as the set of points $\vec{x}$ so that*

   $$\langle \vec{x}, \vec{n} \rangle = c \geq 0$$

   where $\vec{n}$ is a unit vector called the normal vector *to the line and c is a constant.*

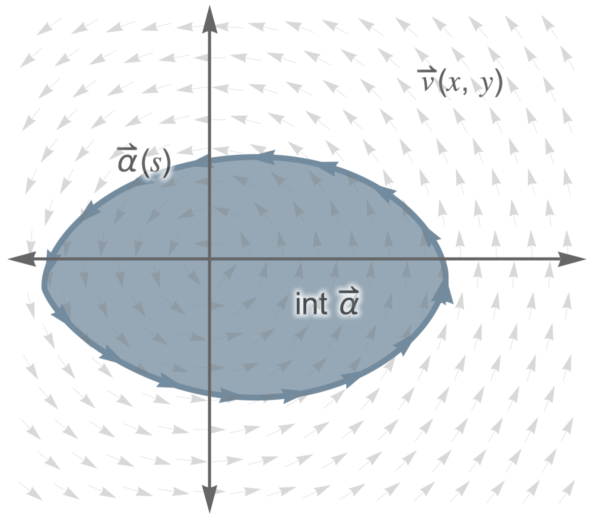
   (1) (5 points) Prove that $\vec{\alpha}'(s) = \vec{m}$ and that $\langle \vec{m}, \vec{n} \rangle = 0$.

   (2) (5 points) Prove that $c = \min_s \|\vec{\alpha}(s)\|$, and that if $\|\vec{\alpha}(s_0)\| = c$, then $\vec{\alpha}(s_0) = \lambda \vec{n}$ for some $\lambda \geq 0$. Although you can do this as a single variable minimization problem, try to solve the problem by Lagrange multipliers (because that generalizes better to higher dimensions).

2. (15 points) Let's remember some multivariable calculus.

Definition. *A map* $\vec{v}\colon \mathbb{R}^2 \to \mathbb{R}^2$ *is called a* vector field. *A vector field is given by two coordinate functions, each a function of two variables:*

$$\vec{v}(x,y) = (v_1(x,y), v_2(x,y))$$



Theorem (Green's Theorem)**.** *If* $\vec{\alpha}(s)\colon S^1 \to \mathbb{R}^2$ *is a closed curve with length* L *and no self-intersections which is oriented counterclockwise,* $\operatorname{int}\vec{\alpha}$ *is the region inside the curve, and* $\vec{v}(x,y)$ *is a (*C^1*) vector field on* $\mathbb{R}^2$ *as in the figure above, then*

$$\int_{\operatorname{int}\vec{\alpha}} \frac{\partial}{\partial x} v_2(x,y) - \frac{\partial}{\partial y} v_1(x,y)\, dx\, dy = \int_0^L \langle \vec{v}(\vec{\alpha}(s)), \vec{\alpha}'(s)\rangle \; ds$$

Green's theorem is a special case of Stokes' theorem, which applies to vector fields in 3-dimensional space. We'll use Stokes' theorem when we study surface theory in the next part of the course!

(1) (10 points) Prove the following:

Proposition. *If $\vec{\alpha}(s)$ is a closed curve with length L and no self-intersections which is oriented counterclockwise and* $\operatorname{int}\vec{\alpha}$ *is the region inside the curve,*

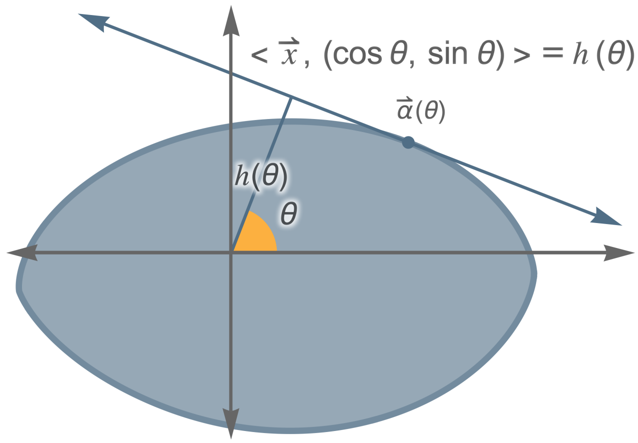
$$\operatorname{area}\operatorname{int}\vec{\alpha} = \frac{1}{2}\int_0^L \left\langle \vec{\alpha}(s)^{\perp}, \vec{\alpha}'(s) \right\rangle \, ds$$

Hint: Use Green's theorem.

(2) (5 points) Suppose that $\vec{\beta}(t) = \vec{\alpha}(s(t))$ is some reparametrization of $\vec{\alpha}(s)$ by a function $s(t)$ with $s(0) = 0$ and $s(1) = L$. Prove the (somewhat surprising!) result that

$$\int_0^L \left\langle \vec{\alpha}(s)^\perp, \vec{\alpha}'(s) \right\rangle \, ds = \int_0^1 \left\langle \vec{\beta}(t)^\perp, \vec{\beta}'(t) \right\rangle \, dt$$

3. (40 points) Suppose that $\vec{\alpha}(t)\colon\ \mathbb{R} \to \mathbb{R}^2$ is a convex[a] curve with the origin inside the curve. There is exactly one tangent line to $\vec{\alpha}$ given in implict form by $\langle \vec{x}, (\cos\theta, \sin\theta)\rangle = h(\theta) > 0$ for each angle θ.



Definition. *We call* $h(\theta)$ *the* support function *of the curve* $\vec{\alpha}(\theta)$.

(1) (10 points) Prove that the curve $\vec{\alpha}(\theta)$ can be expressed in terms of $h(\theta)$ as

$$\vec{\alpha}(\theta) = (h(\theta)\cos\theta - h'(\theta)\sin\theta, h(\theta)\sin\theta + h'(\theta)\cos\theta).$$

using the hypothesis that $\langle \vec{x}, (\cos\theta, \sin\theta)\rangle = h(\theta)$ is the tangent line to $\vec{\alpha}(\theta)$.

[a]Like a convex function, a convex curve has the property that the chord joining any two points on the curve is contained inside the curve. Equivalently, the signed curvature $\kappa_{\pm}(t) > 0$ for all t.

(2) (10 points) Prove that the curvature of the curve at $\vec{\alpha}(\theta)$ is given by

$$\kappa_{\pm}(\theta) = \frac{1}{h(\theta) + h''(\theta)}$$

Hint: Since the curve $\vec{\alpha}(\theta)$ is not parametrized by arclength, you'll need the general formula for $\kappa_{\pm}(t)$ for non-arclength parametrized curves from the last homework.

(3) (10 points) Prove that the length of the curve is given by

$$\operatorname{len}(\vec{\alpha}) = \int_0^{2\pi} h(\theta)\, d\theta.$$

(4) (10 points) Prove that the area inside $\vec{\alpha}$ is given by

$$\operatorname{area}\operatorname{int}\vec{\alpha} = \frac{1}{2}\int_0^{2\pi} (h(\theta)^2 - h'(\theta)^2)\, d\theta.$$

Hint: Use Question 2.

4. (25 points) In the homework on calculus of variations, you proved (with some effort) that the function $y(x)$ with $y(-1) = 0$, $y(1) = 0$ and integral $\int_{-1}^{1} y(x)\,dx$ whose graph had the shortest arclength was the semicircle.

   We are now going to use the support function to prove the stronger theorem that among all convex curves with length 2π, the one which encloses the most area is the circle.

   (1) (5 points) Suppose that $\vec{\alpha}(\theta)$ is a convex curve with support function $h(\theta)$, as in the last question. Write the Lagrangian

   $$\mathcal{L}(\alpha) = \operatorname{area}\operatorname{int}\vec{\alpha} + \lambda \operatorname{len}\vec{\alpha}$$

   in terms of the support function $h(\theta)$ in the form

   $$\mathcal{L}(h) = \int_0^{2\pi} f(h(\theta), h'(\theta))\,d\theta.$$

(2) (10 points) Since the Lagrangian is in the form

$$\mathcal{L}(h) = \int_0^{2\pi} f(h(\theta), h'(\theta))\, d\theta.$$

We know that $h(\theta)$ solves the Euler-Lagrange equation if and only if the Hamiltonian

$$\mathcal{H}(h, h') = f(h, h') - h' \frac{\partial f}{\partial h'}(h, h')$$

is equal to some constant C on the interval $[0, 2\pi]$. Write down an explicit formula for $\mathcal{H}(h, h')$ and use it to find a (first order) ordinary differential equation satisfied by the support function $h(\theta)$ of the largest-area curve.

(3) (10 points) Solve the ODE for $h(\theta)$ you obtained in the last question, using the boundary conditions and the length condition $\mathrm{len}(h) = 2\pi$ to eliminate all unknown constants (including λ). You may assume that we've translated and rotated the curve so that $\vec{\alpha}(0) = \vec{\alpha}(2\pi) = (0,0)$, or $h(0) = h(2\pi) = 0$ and $h'(0) = h'(2\pi) = 0$. Explain why your solution is a unit circle and where it is centered.