Math 4250 Minihomework: Curves and framings

In this minihomework, we'll work with framings on curves.

1. (30 points) Find the Frenet frame $\vec{T}(s)$, $\vec{N}(s)$, $\vec{B}(s)$, $\kappa(s)$ and $\tau(s)$ for the arclength-parametrized curve
$$\vec{\alpha}(s) = \left(\frac{1}{3}(1+s)^{3/2}, \frac{1}{3}(1-s)^{3/2}, \frac{1}{\sqrt{2}}s\right) \quad \text{where} \quad s \in (-1,1).$$
We are going to break this down and do it in an extremely systematic way. The computations involve a bit of algebra[1] but the task should be clear at each step. It's easier to work with your submissions in Gradescope if we put each part on a separate page, so you might have lots of space left over in some of these boxes.
   (1) (5 points) Find the tangent vector $T(s)$ using the formula $T(s) = \vec{\alpha}'(s)$. Check your work by verifying that $\langle T(s), T(s)\rangle = 1$.

[1] Which you can do with Mathematica if you want to, just submit screenshots.

(2) (5 points) Find the derivative $T'(s)$.

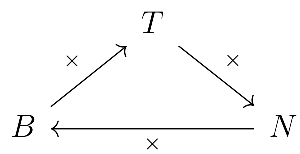
(3) (5 points) Find the curvature $\kappa(s)$ using the formula $\kappa(s) = \|T'(s)\|$. Simplify as much as you can (remember that $s \in (-1, 1)$).

(4) (5 points) Find the normal vector $N(s)$ using the formula $N(s) = \frac{T'(s)}{\kappa(s)}$. Check your work by verifying that $\langle N(s), N(s) \rangle = 1$.

(5) (5 points) Find the binormal vector $B(s)$ using the formula $B(s) = T(s) \times N(s)$. Verify your work by checking that $\langle B(s), B(s) \rangle = 1$.

(6) (5 points) Find the torsion $\tau(s)$ using the formula $\tau(s) = -\langle B'(s), N(s)\rangle$. Be careful to note that this formula involves the *derivative* $B'(s)$ and not $B(s)$ itself. By construction, $\langle B(s), N(s)\rangle = 0$ for all curves.

2. (10 points) (The circle of cross products) Suppose we have three orthonormal vectors T, N, and B in $\mathbb{R}^3$, and that $T \times N = B$. Prove that $N \times B = T$, and that $B \times T = N$. This is often written as the diagram



where products "follow the arrows" clockwise. Going in the counterclockwise direction (i.e. computing $T \times B$, or $B \times N$ or $N \times T$) gives a result with the opposite sign, as the cross product $V \times W = -W \times V$.

Hint: You might find the "bac-cab" identity from "Scalar and Vector Products" helpful.

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) = \vec{b}\,\langle \vec{a}, \vec{c} \rangle - \vec{c}\left\langle \vec{a}, \vec{b} \right\rangle .$$

3. (10 points) (The Darboux Vector) If $\gamma(s)$ is an arclength-parametrized curve with nonzero curvature, find a vector $\omega(s)$, expressed as a linear combination of T, N, and B so that

$$T'(s) = \omega(s) \times T(s)$$
$$N'(s) = \omega(s) \times N(s)$$
$$B'(s) = \omega(s) \times B(s)$$

This vector is called the *Darboux vector*. Find a formula for the length of the Darboux vector in terms of the curvature $\kappa(s)$ and torsion $\tau(s)$ of the curve.

Hint: Any vector $\omega(s)$ can be written as a linear combination of the vectors $T(s)$, $N(s)$, and $B(s)$ with coefficients $a(s)$, $b(s)$ and $c(s)$ which are (scalar) functions of s because $T(s)$, $N(s)$ and $B(s)$ always form an orthonormal basis for $\mathbb{R}^3$ (regardless of s). That is,

$$\vec{\omega}(s) = a(s)\vec{T}(s) + b(s)\vec{N}(s) + c(s)\vec{B}(s).$$

So really the problem is to figure out the functions $a(s)$, $b(s)$ and $c(s)$.

4. (10 points) (Framing plane curves) If $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^2$, we may use the perp operator[2] from the "Square-Wheeled car" homework to define a frame:

$$T(s) = \vec{\alpha}'(s), \quad N(s) = T(s)^{\perp}.$$

Definition. *If $\vec{\alpha}(s)$ is a plane curve, the* signed[3] curvature *is* $\kappa_{\pm}(s) := \langle T'(s), N(s) \rangle$.

Suppose that $\vec{\alpha}(s) = (r\cos\frac{s}{r}, r\sin\frac{s}{r})$ is the circle of radius r (parametrized counterclockwise), and $\vec{\beta}(s) = (r\cos\frac{s}{r}, -r\sin\frac{s}{r})$ is the circle of radius r (parametrized clockwise). Find the signed curvature $\kappa(s)$ for each curve.

[2] Remember that $(x, y)^{\perp} = (-y, x)$.

[3] Notice that unlike the curvature $\kappa(s)$ for space curves, which is equal to $\|T'(s)\|$ and hence always non-negative, the signed curvature $\kappa_{\pm}(s)$ can have either sign because the dot product which defines it can be negative.