Math 4250 Minihomework: Curvature and Torsion Theorems

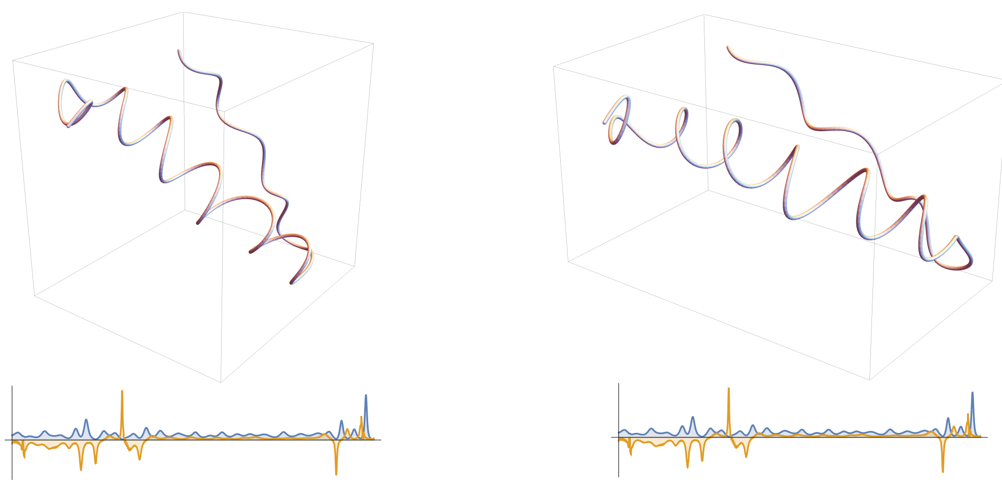
Definition. *If* $A \in \mathrm{SO}(n)$ *and* $\vec{v} \in \mathbb{R}^n$*, we say that the map* $\vec{x} \mapsto A\vec{x} + \vec{v}$ *is a* rigid motion. *The rigid motions form a group called the* special Euclidean group $\mathrm{SE}(n)$.

Rigid motions preserve angles, lengths, and handedness.

Definition. *We say that parametrized curves* $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^n$ *and* $\vec{\beta}\colon \mathbb{R} \to \mathbb{R}^n$ *are* congruent *if* $\vec{\alpha}(t)$ *and* $\vec{\beta}(t)$ *are related by a rigid motion. That is,* $\vec{\alpha}(t)$ *and* $\vec{\beta}(t)$ *are congruent iff there exists some* $A \in \mathrm{SO}(n)$ *and* $\vec{v} \in \mathbb{R}^n$ *so that* $\vec{\beta}(t) = A\vec{\alpha}(t) + \vec{v}$ *for all* t.

Theorem (Fundamental Theorem of Curve Theory)**.** *Two space curves* $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ *and* $\vec{\beta}\colon \mathbb{R} \to \mathbb{R}^3$ *with nonzero curvature are congruent if and only if the curvature* $\kappa^\alpha(t) = \kappa^\beta(t)$ *and torsion* $\tau^\alpha(t) = \tau^\beta(t)$ *for all* t.

This theorem implies that we can identify congruent curves by comparing curvatures and torsions. For example, the two curves below are based on the geometry of the backbone of the protein 1ppt (avian pancreatic polypeptide). Though they may not look the same, graphing their curvature (blue) and torsion (orange) reveals that they are actually congruent.



We are now going to prove the fundamental theorem together by completing a set of problems. We'll need to recall some ideas from the "Tale of Two Matrices" video lecture:

Definition. *An* $n \times n$ *matrix* A *is* orthogonal *if* $AA^T = I_n$*. In this case, we say that* A *is a member of the* orthogonal group $\mathrm{O}(n)$*. Every orthogonal matrix has* $\det A = \pm 1$*. If* A *is orthogonal and* $\det A = +1$*, we say that* A *is a member of the* special orthogonal group $\mathrm{SO}(n)$.

Proposition. *If* $A \in \mathrm{SO}(3)$*, then* A *is a rotation around some axis.*

1. (8 points) You proved in the "Getting Comfortable again with Linear Algebra" homework that $\mathrm{O}(n)$ is a group[a]: that is,

Proposition. *If A and B are orthogonal matrices, then AB is an orthogonal matrix. If A is an orthogonal matrix, then A^{-1} is an orthogonal matrix.*

Assuming that $\mathrm{O}(n)$ is a group (or equivalently that the proposition directly above is true), you will now show that $\mathrm{SO}(n)$ is a subgroup of $\mathrm{O}(n)$.[b] There are two things that you must prove:

(1) (4 points) Show that if A and B are in $\mathrm{SO}(n)$, then AB is also in $\mathrm{SO}(n)$.

(2) (4 points) Show that if A is in $\mathrm{SO}(n)$ then A^{-1} is also in $\mathrm{SO}(n)$.

[a] In fact, it's a subgroup of the group $\mathrm{GL}(n)$ of $n \times n$ invertible matrices.

[b] In fact, $\mathrm{SO}(n)$ is both a subgroup of $\mathrm{O}(n)$ and of the group $\mathrm{SL}(n)$ of $n \times n$ invertible matrices with positive determinant (which is itself a subgroup of $\mathrm{GL}(n)$).

2. (25 points) Recall from the notes that we claimed

Theorem. *If* $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ *is an arclength-parametrized curve framed by* $F\colon \mathbb{R} \to \mathrm{SO}(3)$*,* $A \in \mathrm{SO}(3)$*, and* $\vec{v} \in \mathbb{R}^3$*, then* $\vec{\beta}(s) = A\vec{\alpha}(s) + \vec{v}$ *is an arclength-parametrized curve framed by* $AF\colon \mathbb{R} \to \mathrm{SO}(3)$.

You are now going to prove this theorem.

(1) (5 points) We start by proving a new version[c] of the product rule. Suppose that $A\colon \mathbb{R} \to \mathrm{Mat}_{n\times k}(\mathbb{R})$[d] and $B\colon \mathbb{R} \to \mathrm{Mat}_{k\times m}(\mathbb{R})$ are matrix-valued functions. Prove that

$$\frac{d}{dt}A(t)\cdot B(t) = A'(t)\cdot B(t) + A(t)\cdot B'(t)$$

where $\cdot$ is matrix multiplication. Be sure to clearly indicate where you use the ordinary product rule $\frac{d}{dt}f(t)g(t) = f'(t)g(t) + f(t)g'(t)$. Hint: Look up the definition of matrix multiplication and prove the formula entry-by-entry.

[c] So far, we have proved two different versions of the product rule for vector-valued functions $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^n$ and $\vec{\beta}\colon \mathbb{R} \to \mathbb{R}^n$: We have $\frac{d}{dt}\left\langle \vec{\alpha}(t), \vec{\beta}(t)\right\rangle = \left\langle \vec{\alpha}'(t), \vec{\beta}(t)\right\rangle + \left\langle \vec{\alpha}(t), \vec{\beta}'(t)\right\rangle$. Further, if $n = 3$, then we have $\frac{d}{dt}\vec{\alpha}(t) \times \vec{\beta}(t) = \vec{\alpha}'(t) \times \vec{\beta}(t) + \vec{\alpha}(t) \times \vec{\beta}'(t)$. You can expect that the proof of this product rule is pretty much the same.

[d] This is the space of $n \times k$ matrices with real entries.

(2) (5 points) Suppose that $A\colon \mathbb{R} \to \mathrm{Mat}_{n\times m}(\mathbb{R})$ and $B\colon \mathbb{R} \to \mathrm{Mat}_{n\times m}(\mathbb{R})$. Prove that

$$\frac{d}{dt}\left(A(t) + B(t)\right) = A'(t) + B'(t)$$

Hint: As before, refer to the definition of matrix addition and prove the formula entry-by-entry.

(3) (5 points) Suppose that $\vec{\alpha}(s)$ is a unit speed curve, $A \in \mathrm{SO}(3)$ and $\vec{v} \in \mathbb{R}^3$, and let $\vec{\beta}(s) = A\vec{\alpha}(s) + \vec{v}$. Prove that $\vec{\beta}'(s) = A\vec{\alpha}'(s)$ (2.1 and 2.2 will help) and then prove that $\vec{\beta}(s)$ is also a unit speed curve. You'll need to use the fact that A is an orthogonal matrix.

(4) (10 points) Prove that if $F(s)$ is a framing[e] of the unit-speed curve $\vec{\alpha}(s)$, $A \in \mathrm{SO}(3)$ and $\vec{v} \in \mathbb{R}^3$, then $AF(s)$[f] is a framing of the unit-speed curve $\vec{\beta}(s) = A\vec{\alpha}(s) + \vec{v}$.

Hint: You must show that $A \cdot F(s) \in \mathrm{SO}(3)$ for all s and that the first column of $AF(s)$ is the tangent vector $T^{\beta}(s)$ to $\vec{\beta}(s)$. The previous questions will help; remember that matrix-vector multiplication is the special case of matrix multiplication where the second matrix is a column vector.

[e] Recall from the notes that

Definition. *A framing of* $\vec{\alpha}\colon \mathbb{R} \to \mathbb{R}^3$ *is a map* $F\colon \mathbb{R} \to \mathrm{SO}(3)$ *so that*

$$F(s) = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T(s) & F_1(s) & F_2(s) \\ \downarrow & \downarrow & \downarrow \end{pmatrix}$$

where $T(s) = \vec{\alpha}'(s)$.

[f] The matrix product of A and $F(s)$.

3. (25 points) Recall our proposition[g]

Proposition. *If* $F\colon \mathbb{R} \to \mathrm{SO}(3)$ *is a framing of* $\vec{\alpha}(s)$*, then*

$$F'(s) = F(s)\mathcal{S}(s)$$

where $\mathcal{S}(s)$ *is skew-symmetric.*[h]

Suppose that $\vec{\alpha}(s)$ is a unit-speed curve framed by $F\colon \mathbb{R} \to \mathrm{SO}(3)$, $A \in \mathrm{SO}(3)$ and $\vec{v} \in \mathbb{R}^3$. Let $\vec{\beta}(s) = A\vec{\alpha}(s) + \vec{v}$. In the last problem, you proved that $\vec{\beta}(s)$ is a unit-speed curve framed by $AF\colon \mathbb{R} \to \mathrm{SO}(3)$. Therefore, there are skew-symmetric matrices $\mathcal{S}^\alpha(s)$ and $\mathcal{S}^\beta(s)$ so that

$$F'(s) = F(s)\mathcal{S}^\alpha(s)$$
$$(AF)'(s) = (AF)(s)\mathcal{S}^\beta(s)$$

(1) (10 points) Prove that $\mathcal{S}^\alpha(s) = \mathcal{S}^\beta(s)$.

[g] From the notes on Framings, page 5.
[h] That is, $\mathcal{S}(s)^T = -\mathcal{S}(s)$.

(2) (10 points) We start by recalling a few facts:

Definition. *The Frenet frame for a unit-speed curve $\vec{\alpha}(s)$ is given by*

$$T(s) = \vec{\alpha}'(s), \qquad N(s) = \frac{\vec{\alpha}''(s)}{\|\vec{\alpha}''(s)\|}, \quad B(s) = T(s) \times N(s).$$

The usual form of the Frenet equations is

$$\begin{array}{llll} T'(s) & & +\kappa(s)N(s) & \\ N'(s) = & -\kappa(s)T(s) & & +\tau(s)B(s) \\ B'(s) & & -\tau(s)N(s) & \end{array}$$

The Frenet equations can also be written as the matrix equation (note the signs!)

$$\begin{aligned} F'(s) &= \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T'(s) & N'(s) & B'(s) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \\ &= \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ T(s) & N(s) & B(s) \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} = F'(s)\mathcal{S}(s). \end{aligned}$$

Suppose that $F\colon \mathbb{R} \to \mathrm{SO}(3)$ is a framing of a unit speed curve $\vec{\alpha}(s)$.

Prove that F is the Frenet frame $\iff$ we have $F'(s) = F(s)\mathcal{S}(s)$ for a skew-symmetric matrix $\mathcal{S}(s)$ with $\mathcal{S}(s)_{21} > 0$, $\mathcal{S}(s)_{32} > 0$, and $\mathcal{S}(s)_{13} = -\mathcal{S}(s)_{31} = 0$.

(3) (5 points) Using the two previous parts of this question and the matrix form of the Frenet equations, show that if $\vec{\alpha}(s)$ is a unit-speed curve, $A \in \mathrm{SO}(3)$ and $\vec{v} \in \mathbb{R}^3$, and $\vec{\beta}(s) = A\vec{\alpha}(s) + \vec{v}$, then $\vec{\alpha}(s)$ and $\vec{\beta}(s)$ have the same curvature $\kappa^{\alpha}(s) = \kappa^{\beta}(s)$ and torsion $\tau^{\alpha}(s) = \tau^{\beta}(s)$ at each s.

4. (15 points) Suppose that $\vec{\alpha}(s)$ and $\vec{\beta}(s)$ are arclength-parametrized curves with nonvanishing curvature, with Frenet frames $T_\alpha(s), N_\alpha(s), B_\alpha(s)$ and $T_\beta(s), N_\beta(s), B_\beta(s)$. Suppose that

$$T_\alpha(0) = T_\beta(0), \qquad N_\alpha(0) = N_\beta(0), \quad \text{and} \quad B_\alpha(0) = B_\beta(0)$$

and that $\vec{\alpha}(0) = \vec{\beta}(0)$. Further, suppose that the curvature $\kappa^\alpha(s) = \kappa^\beta(s)$ and torsion $\tau^\alpha(s) = \tau^\beta(s)$ are equal for all s.

(1) (10 points) Prove that

$$T_\alpha(s) = T_\beta(s), \qquad N_\alpha(s) = N_\beta(s), \quad \text{and} \quad B_\alpha(s) = B_\beta(s) \qquad (\bigstar)$$

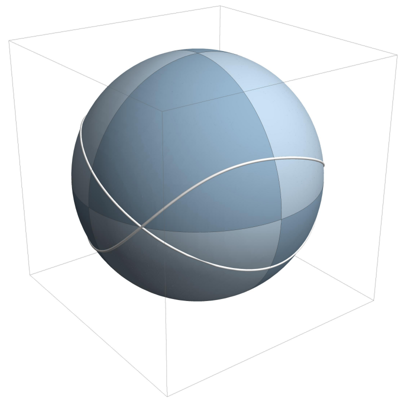
for all s. Here is a strategy to follow. First, consider the function

$$f(s) = \|T_\alpha(s) - T_\beta(s)\|^2 + \|N_\alpha(s) - N_\beta(s)\|^2 + \|B_\alpha(s) - B_\beta(s)\|^2$$

Notice that equation ($\bigstar$) holds if and only if $f(s) = 0$ for all s. Prove that $f(s) = 0$ for all s by proving that $f(0) = 0$ and $f'(s) = 0$ for all s.

(2) (5 points) You have just proved (in particular) that $\vec{\alpha}'(s) = T^{\alpha}(s) = T^{\beta}(s) = \vec{\beta}'(s)$ for all s. Use this and the fact that $\vec{\alpha}(0) = \vec{\beta(0)}$ to show that $\vec{\alpha}(s) = \vec{\beta}(s)$ for all s.

5. Suppose that we have a curve $\vec{\alpha}(s)$ which lies on the unit sphere, as below:



It seems reasonable that such a curve should have some restriction on its curvature and torsion.

(1) (10 points) Suppose that $\vec{\alpha}(s)$ lies on a sphere of radius R centered at the origin. Prove that $\kappa(s) \neq 0$ and either ($\tau(s) = 0$ and $\kappa(s)$ is constant) or

$$\tau(s) \neq 0 \quad \text{and} \quad \frac{\tau(s)}{\kappa(s)} + \frac{d}{ds}\left(\frac{1}{\tau(s)} \cdot \frac{d}{ds}\frac{1}{\kappa(s)}\right) = 0.$$

Without loss of generality, you may assume that $\vec{\alpha}(s)$ is unit-speed.

Hint: Because $T(s), N(s), B(s)$ is an orthonormal basis for $\mathbb{R}^3$, you can *always* write

$$\vec{\alpha}(s) = \langle\vec{\alpha}(s), T(s)\rangle\, T(s) + \langle\vec{\alpha}(s), N(s)\rangle\, N(s) + \langle\vec{\alpha}(s), B(s)\rangle\, B(s)$$

Start with the equation $R^2 = \langle\vec{\alpha}(s), \vec{\alpha}(s)\rangle$ and keep differentiating and simplifying both sides[i] until you can write $\langle\vec{\alpha}(s), T(s)\rangle$, $\langle\vec{\alpha}(s), N(s)\rangle$, and $\langle\vec{\alpha}(s), B(s)\rangle$ in terms of $\kappa(s)$ and $\tau(s)$. Then differentiate one more time.

[i] Use the Frenet equations and the fact that $\vec{\alpha}'(s) = T(s)$ as needed.

(2) (10 points) Suppose that

$$\kappa(s) \neq 0 \quad \text{and} \quad \frac{\tau(s)}{\kappa(s)} + \frac{d}{ds}\left(\frac{1}{\tau(s)} \cdot \frac{d}{ds}\frac{1}{\kappa(s)}\right) = 0. \qquad (\bigstar)$$

Prove that $\vec{\alpha}(s)$ lies on the surface of some[j] sphere.

Hint: The really hard part in this question is determining where the sphere is centered. A savvy guess would be to define

$$\mathcal{C}(s) = \vec{\alpha}(s) - a(s)T(s) - b(s)N(s) - c(s)B(s)$$

where $a(s)$, $b(s)$ and $c(s)$ are expressed in terms of $\kappa(s)$ and $\tau(s)$ by the formulae from the previous part. In general, we expect $\mathcal{C}(s)$ to be some weird new space curve.

Show that if ($\bigstar$) holds, then $\mathcal{C}'(s) = 0$ and $\|\vec{\alpha}(s) - \mathcal{C}\|$ is constant, so $\vec{\alpha}(s)$ lies on a sphere centered at $\mathcal{C}$.

[j] The sphere does *not* have to be centered at the origin!