Math/Csci 4690/6690 : Computations with the graph Laplacian, isomorphisms, drawings

In this minihomework, you'll get some practice using computational tools to work with small graphs. Here are edge lists for three graphs, each with 10 vertices $\{1, \ldots, 10\}$ and 23 edges:

$$G_1 = \{1 \leftrightarrow 3, 1 \leftrightarrow 5, 1 \leftrightarrow 9, 2 \leftrightarrow 3, 2 \leftrightarrow 4, 2 \leftrightarrow 5, 2 \leftrightarrow 6, 2 \leftrightarrow 10, 3 \leftrightarrow 4, 3 \leftrightarrow 6, 3 \leftrightarrow 10, 4 \leftrightarrow 6, 4 \leftrightarrow 8, 4 \leftrightarrow 9, 4 \leftrightarrow 10, 5 \leftrightarrow 6, 5 \leftrightarrow 9, 6 \leftrightarrow 7, 6 \leftrightarrow 8, 6 \leftrightarrow 9, 7 \leftrightarrow 9, 8 \leftrightarrow 9, 8 \leftrightarrow 10\}$$

$$G_2 = \{1 \leftrightarrow 2, 1 \leftrightarrow 3, 1 \leftrightarrow 4, 1 \leftrightarrow 5, 2 \leftrightarrow 3, 2 \leftrightarrow 6, 2 \leftrightarrow 10, 3 \leftrightarrow 4, 3 \leftrightarrow 5, 3 \leftrightarrow 6, 3 \leftrightarrow 7, 3 \leftrightarrow 8, 3 \leftrightarrow 9, 4 \leftrightarrow 7, 4 \leftrightarrow 8, 5 \leftrightarrow 6, 5 \leftrightarrow 7, 6 \leftrightarrow 7, 6 \leftrightarrow 8, 6 \leftrightarrow 9, 6 \leftrightarrow 10, 8 \leftrightarrow 10, 9 \leftrightarrow 10\}$$

$$G_3 = \{5 \leftrightarrow 3, 5 \leftrightarrow 6, 5 \leftrightarrow 2, 4 \leftrightarrow 3, 4 \leftrightarrow 7, 4 \leftrightarrow 6, 4 \leftrightarrow 9, 4 \leftrightarrow 8, 3 \leftrightarrow 7, 3 \leftrightarrow 9, 3 \leftrightarrow 8, 7 \leftrightarrow 9, 7 \leftrightarrow 10, 7 \leftrightarrow 2, 7 \leftrightarrow 8, 6 \leftrightarrow 9, 6 \leftrightarrow 2, 9 \leftrightarrow 1, 9 \leftrightarrow 10, 9 \leftrightarrow 2, 1 \leftrightarrow 2, 10 \leftrightarrow 2, 10 \leftrightarrow 8\}$$

Recall that in the notes, we gave

Definition. *Given two graphs G and G', a map $f\colon \{v_1, \ldots, v_{\mathbf{v}}\} \to \{v'_1, \ldots, v'_{\mathbf{v}'}\}$ is an* isomorphism *between graphs if*

1. *f is a bijection between the vertices of G and the vertices of G',*
2. *the map $v_i \leftrightarrow v_j \to f(v_i) \leftrightarrow f(v_j)$ is a bijection between the edges of G and the edges of G'.*

Notice that if the two graph have different numbers of vertices or edges, they cannot be isomorphic. Further, any bijection f is equivalent to a permutation π of the indices of the vertices: there must exist some π so that $f(v_i) = v'_{\pi(i)}$.

Some of these problems require a modest amount of computation. I've added some optional links on the course webpage to helpful functions if you'd like to do these in Python (in which case I recommend the iGraph python bindings) or if you'd like to do these in Mathematica. You're welcome to use other languages as well.

1. (10 points) Show that if G and G' are isomorphic, there exists some permutation matrix Π so that

$$L_G = \Pi L_{G'} \Pi^T$$

Use the last homework to conclude that L_G and $L_{G'}$ have the same spectrum (set of eigenvalues).

2. (10 points) Find the graph Laplacian for each of the three graphs above. In each case, this should be a symmetric 10×10 matrix with 23 (-1) entries in the upper triangle, 23 (-1) entries in the lower triangle, and 10 nonzero entries on the main diagonal.

3. (10 points) Use your favorite computer program or website to find the eigenvalues (the spectrum) of each of the graph Laplacian matrices you found in 2, and provide a list of the eigenvalues.

   Use the result of 1 to decide whether it may be possible to find an isomorphism between G_1 and G_2, or between G_2 and G_3, or between G_1 and G_3, and write down your decision and a description of your reasoning.

   Be sure to clearly distinguish between cases where the existence of an isomorphism is ruled out, cases where the existence of an isomorphism is guaranteed, and cases where the existence of an isomorphism is unknown.

4. (15 points) The eigen*vectors* of a matrix are determined only up to sign and scale. However, if you compute an eigenvector of a graph Laplacian, rescale it to have unit norm, and choose signs to make the first entry positive, you'll discover that the entries in the eigenvector are the same for corresponding vertices in a pair of isomorphic graphs. This determines an isomorphism uniquely unless two components of the eigenvector are the same (in which case, you can see if another eigenvector tells you what to do with those!).

   (1) (5 points) Use your favorite computer program or website to find the eigenvectors of L_G for any pair of graphs which you concluded above might be isomorphic.

(2) (5 points) Now use the entries in the eigenvectors to match up pairs of vertices in your two graphs and define a proposed isomorphism f between them.

(3) (5 points) Check that your proposed isomorphism really does take the edge set of one graph to the edge set of the other.

5. (10 points) (For students in CSCI 4690, CSCI 6690, and MATH 6690). Use your favorite programming language or software package to create drawings of all three graphs using the eigenvector with eigenvalue λ_2 to assign x coordinates and the eigenvector with eigenvalue λ_3 to assign y coordinates.[1]

[1] Note that these are the second and third *smallest* eigenvalues, and that eigenvectors may be given in a different order depending on your computational system.