Math 6500 Minihomework: Bernoulli numbers and polynomials

This minihomework covers our notes on the advanced error analysis of the trapezoid rule. We'll make a lot of use of the two formulae:

$$\frac{t}{e^t - 1} = \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \tag{1}$$

and

$$\frac{t(e^{xt} - 1)}{e^t - 1} = \sum_{j=0}^{\infty} B_j(x) \frac{t^j}{j!}. \tag{2}$$

1. In the demonstration, we saw that the Bernoulli numbers and polynomials were related by the identity
$$B_i = -\int_0^1 B_i(x)\,dx$$
for $i > 1$. We are now going to prove this in several steps.

   a. Integrate both sides of (2) with respect to x from 0 to 1 to get a new series where the coefficients are integrals of $B_j(x)$.

   b. Prove using (1) that these integrals are actually the Bernoulli numbers B_j, except at $j = 0$.

2. In the notes, we gave as an exercise the task of proving that
$$\frac{d}{dx} B_j(x) = j B_{j-1}(x), \qquad \text{for } j \geq 4,\ j \text{ even}$$
and
$$\frac{d}{dx} B_j(x) = j\left(B_{j-1}(x) + B_{j-1}\right) \qquad \text{for } j \geq 3,\ j \text{ odd}.$$

   Prove both of these formulae. Hint: Experiment with differentiating both sides of (2) with respect to x.