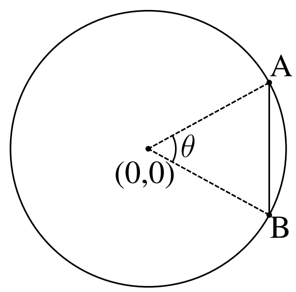
Math 4250 Minihomework: Reparametrizing curves by arclength

In this minihomework, we'll work out some important ideas about length for curves. This will give us a chance to establish a general framework that we'll use to prove theorems about curve length later in the course.

1. (15 points) Recall that our definition of the length of a curve is that

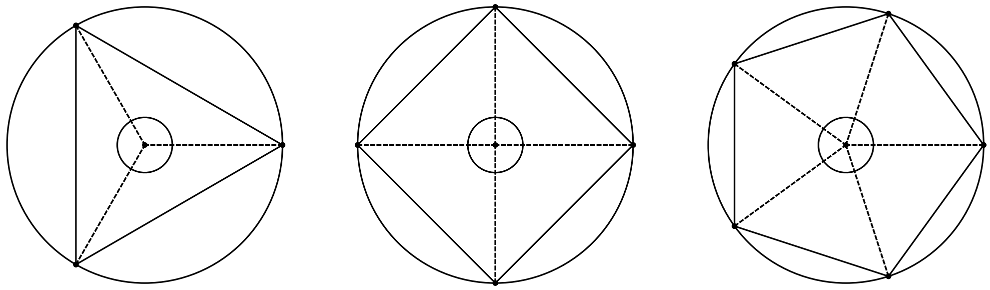
$$\operatorname{length}(C) = \sup_{\substack{P \text{ is a polygon} \\ P \text{ is inscribed in } C}} \operatorname{length}(P).$$

We're now going to prove *directly* that the circumference of the unit circle is 2π.



(1) (5 points) Find a formula for the length of the dotted chord AB in terms of the angle θ using trigonometry. Give a proof that your formula is correct. Check your formula by making sure that the length is 2 when $\theta = \pi$.

(2) (5 points) Suppose that P_n is an inscribed equilateral n-gon. The pictures below show the situation for $n = 3$, 4, and 5:



Find the length of each of these polygons as a function of n by writing each side as a chord of the same angle θ_n[1], computing the length of each chord using your result for 1.1, and summing the results.[2]

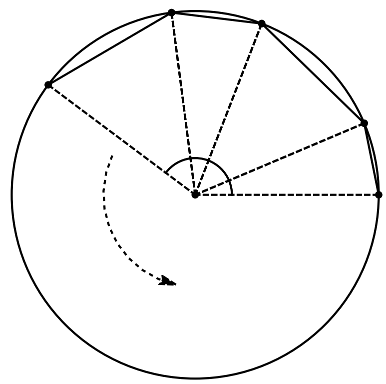
Now prove that the limit of these lengths as the number of sides $n \to \infty$ is 2π. This provides a lower bound on the length of the circle.[3]

[1]Fun fact: This is called the angle *subtended* by the chord.

[2]In particular, you *don't* have to use the definition of length as the $\sup$ of lengths of inscribed polygons to compute the length of each of these polygons.

[3]In particular, you've proved that for every number $m < 2\pi$, there's some regular inscribed polygon with length $> m$. This means that no number smaller than 2π could be the least upper bound bound on the lengths of all inscribed polygons, and hence that the length of the circle (the least upper bound on lengths of inscribed polygons) is at least 2π.

(3) (5 points) Now suppose that we consider a general polygon P_n inscribed in the circle as below, where the angles $\theta_1, \ldots, \theta_n$ still have $\theta_1 + \cdots + \theta_n = 2\pi$, but the θ_i are not all the same.



Using the fact (for $\theta \geq 0$) that $\sin\theta \leq \theta$, prove that the sum of the lengths of the sides of this polygon must be $\leq 2\pi$. This proves that 2π is an upper bound for the length of any inscribed polygon and hence an upper bound on the length of the circle.[4]

Combined with the results of the last question, you've proved that the length of the circle (according to our definition for rectifiable curves) is exactly 2π.

[4] Because $\sup \operatorname{length} P_n$ is the *least* upper bound on the lengths of inscribed polygons, it must be $\leq$ the particular upper bound of 2π that you've just established.

(4) (5 points) (Extra credit question). Define $\mathrm{mesh}(P_n) = \max_i \theta_i$ to be the largest angle in the subdivision of the circle. Prove that if we have a family of inscribed polygons in the circle with $\lim_{n\to\infty} \mathrm{mesh}(P_n) = 0$, then $\lim_{n\to\infty} \mathrm{length}\, P_n = 2\pi$.

Remember to use the chord length formula you found in part 1 to find the length of the polygon, and recall this helpful inequality (from MATH 2260). For $\theta > 0$,

$$\theta - \frac{\theta^3}{3!} \leq \sin\theta \leq \theta$$

2. (5 points) The circle can be written as a parametrized curve $\vec{\alpha}(t) = (\cos t, \sin t)$. Use our theorem

$$\text{length}\,\vec{\alpha}(t) = \int_a^b \|\vec{\alpha}'(t)\|\,dt$$

to find the circumference of the circle by integration.

3. (15 points) In our notes, we saw

Theorem. *If $\vec{\alpha}(t)$ is a regular curve, then there exists some differentiable function $t(s)$ so that $\vec{\alpha}(t(s))$ is parametrized by arclength.*

We also saw that if

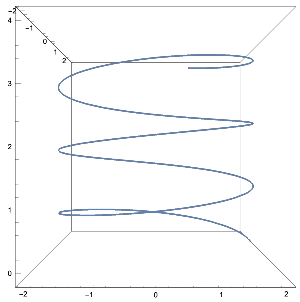
$$s(t) = \int_a^t \|\vec{\alpha}'(t)\|\,dt \tag{1}$$

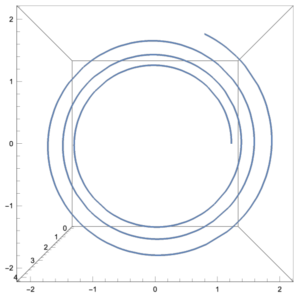
is the arclength function, then $t(s)$ is the inverse of the arclength function. Therefore, in order to explicitly reparametrize a curve $\vec{\alpha}(t)$ by arclength, we need to

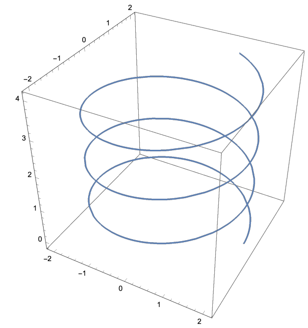
- be able to do the integral in (1),
- be able to find an explicit inverse for the resulting function.

As you might imagine, these cases are fairly rare and specific. However, it's still worth practicing doing this when we can. A couple of notes can help. First, the initial value a in the integral for $s(t)$ is arbitrary; it's usually chosen to simplify the formula for $s(t)$ as much as possible. Second, the "t" inside the integral is a dummy variable, which is different than the "t" in the limits of integration. We could have rewritten (1) as $s(t) = \int_a^t \|\vec{\alpha}'(x)\|\,dx$.

(1) (5 points) Let $\vec{\alpha}(t) = (a\cos t, a\sin t, bt)$ be the helix





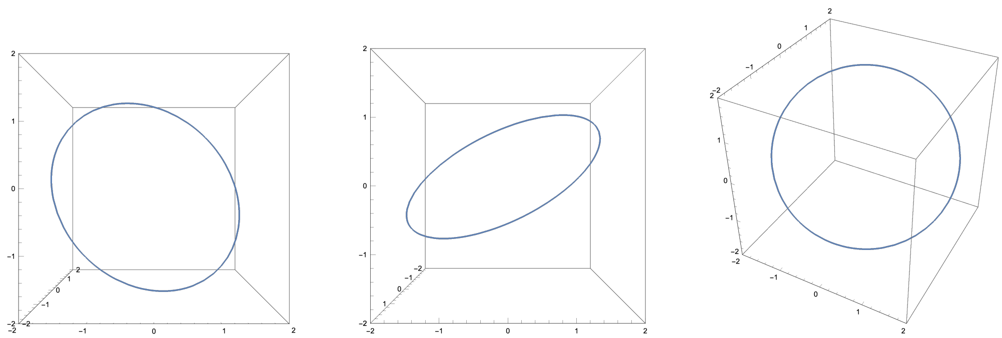


Find $s(t)$ and $t(s)$ and reparametrize $\vec{\alpha}(t)$ by arclength. Give a formula for the arclength-parametrized curve $\vec{\alpha}(s)$ which has the property that $\vec{\alpha}(s) = \vec{\alpha}(t)$ when $s = 0$ and $t = 0$.

(2) (5 points) Let

$$\vec{\alpha}(t) = \left(\frac{2}{\sqrt{3}} \cos t + \frac{2}{\sqrt{2}} \sin t, \frac{2}{\sqrt{3}} \cos t, \frac{2}{\sqrt{3}} \cos t - \frac{2}{\sqrt{2}} \sin t \right)$$

parametrize the curve



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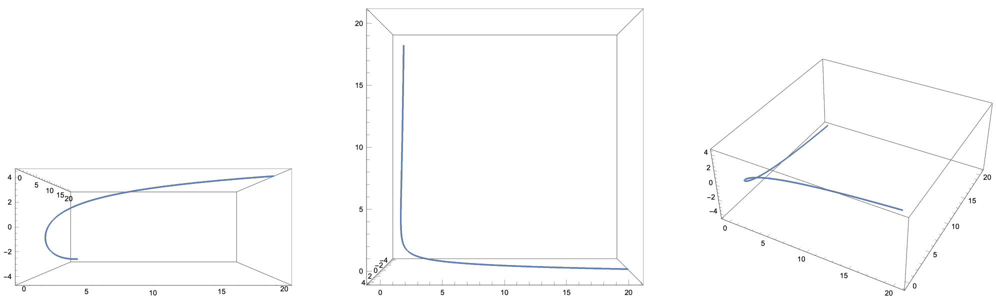
(3) (10 points) Let

$$\vec{\alpha}(t) = \left(e^t, e^{-t}, \sqrt{2}t\right)$$

parametrize the curve



Find $s(t)$ and $t(s)$ and reparametrize $\vec{\alpha}(t)$ by arclength. Give a formula for the arclength-parametrized curve $\vec{\alpha}(s)$ which has the property that $\vec{\alpha}(s) = \vec{\alpha}(t)$ when $s = 0$ and $t = 0$.

Notes:

- The book gives a formula for the final answer on p. 121. However, depending on how you choose to do the algebra, your answer might look different.
- The algebra is not easy. So you should verify that you have the right answer either by checking with a computer algebra system (eg Maple, Mathematica), or by plugging in some numbers to $\|\alpha'(s)\|$ to make sure that the answer is always 1.