Math 4510/6510 Homework #4

1. Let A be an $m \times n$ matrix with $m \geq n$ and suppose A has full rank. Show that the equation
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$
has a solution where x minimizes $||Ax - b||_2$.

2. Assuming as above that A is an $m \times n$ matrix with $m \geq n$ with full rank, what is the condition number of
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$$
in terms of the singular values of A? (Hint: Use the SVD of A.)

3. Assuming as above that A is an $m \times n$ matrix with $m \geq n$ with full rank, find an explicit expression for the inverse of
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix}$$
as a block 2×2 matrix. (Hint: Use 2×2 block Gaussian elimination.)

4. (Bonus) Show how to use the QR decomposition of A to implement an iterative refinement algorithm to improve the accuracy of the solution for x in
$$\begin{pmatrix} I & A \\ A^T & 0 \end{pmatrix} \begin{pmatrix} r \\ x \end{pmatrix} = \begin{pmatrix} b \\ 0 \end{pmatrix}$$

5. Suppose that A is an $m \times n$ matrix with SVD $A = U\Sigma V^T$. Compute the SVDs of the following matrices in terms of U, Σ, and V:
   (1) $(A^TA)^{-1}$
   (2) $(A^TA)^{-1}A^T$
   (3) $A(A^TA)^{-1}$
   (4) $A(A^TA)^{-1}A^T$

6. (Constrained Least Squares) Suppose we want to find x minimizing $||Ax - b||_2$ subject to the linear constraint $Cx = d$. Suppose that A is $m \times n$, C is $p \times n$, and C has full rank. Suppose also that $p \leq n$ (so that we can guarantee that $Cx = d$ has a solution) and $n \leq m + p$ (so that the system is not underdetermined).
   (1) Show that if
   $$\begin{pmatrix} A \\ C \end{pmatrix}$$
   has full column rank, then there is a unique solution.
   (2) (Bonus) Show how to compute the solution x using two QR decompositions, some matrix-vector multiplications, and some solutions of triangular system of linear equations. Hint: Look at the LAPACK routine `sgglse`.