Math 4510/6510 Homework #3

1. Prove the following statements. Let P_1, P_2, P be $n \times n$ permutation matrices and X be any $n \times n$ matrix:
   (1) PX is the same as X with rows permuted. XP is the same as X with columns permuted.
   (2) $P^{-1} = P^T$.
   (3) $\det(P) = \pm 1$.
   (4) $P_1 P_2$ is also a permutation matrix.

2. Write *Mathematica* programs to find the LU decomposition of a matrix with *complete* pivoting and with *partial* pivoting according to the algorithm presented in class. (Hint: No fair using the `LUDecomposition` routine directly, but you should certainly use it to check your results.) The implementation notes on pages 41-44 of Demmel may be helpful.

3. Write a *Mathematica* program which does forward and back substitution. Solve the 5x5 linear system:

$$\begin{pmatrix} -\frac{3}{10} & \frac{1}{10} & \frac{27}{10} & \frac{4}{5} & -\frac{4}{5} \\ -\frac{21}{10} & -2 & \frac{3}{5} & \frac{13}{10} & 1 \\ -\frac{11}{10} & \frac{2}{5} & \frac{1}{5} & -\frac{1}{10} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{10} & \frac{3}{10} & -\frac{1}{5} & \frac{7}{10} \\ \frac{1}{5} & \frac{1}{5} & -\frac{3}{10} & -\frac{13}{10} & \frac{9}{5} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} \frac{2}{5} \\ \frac{7}{10} \\ \frac{8}{5} \\ -\frac{7}{10} \\ 1 \end{pmatrix}$$

   using your partial pivoting and complete pivoting codes from question 2 and the *Mathematica* `LUDecomposition` function. Verify that your answers are close to the correct solution of

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -\frac{241690}{24437} \\ -\frac{308143}{24437} \\ \frac{63503}{24437} \\ -\frac{608283}{24437} \\ -\frac{354063}{24437} \end{pmatrix}$$

   Find the number of correct digits in each (floating point) solution (note that you'll have to multiply the matrix above by 1.0 in order to get *Mathematica* to do the computation in machine arithmetic). Which method is best: partial pivoting, complete pivoting, or *Mathematica*'s `LUDecomposition`?

4. Compare the speed of your programs on small random matrices (50×50, 100×100 and 500×500 matrices) with each other and with the `LUDecomposition` routine built into *Mathematica*. The `Timing` command will be helpful here.

5. Suppose a and b are column vectors. Consider the $n \times n$ matrix ab^T. Prove that this matrix has rank one. Then prove that any rank one matrix A can be written in this form for some a and b.

6. Suppose that A is upper triangular. Prove that the determinant of A is the product of the diagonal entries.

7. Prove that if a dot product is computed in floating point arithmetic then

$$\mathrm{fl}\left(\sum_{i=1}^{d} x_i y_i\right) = \sum_{i=1}^{d} x_i y_i (1+\delta_i) \text{ with } |\delta_i| \leq d\epsilon$$

where ϵ is the machine epsilon for the floating point system used.

8. Suppose two matrices A and B have nonnegative entries and $a_{ij} \leq b_{ij}$ for all i, j in $1 \ldots n$. Prove that $||A|| \leq ||B||$ (in the 2-norm).