Math 4510/6510 Homework #1

1. Derive the 3rd order Taylor rule for solving $x'(t) = f(t, x)$ numerically.
2. Work out the ODE method called *Heun's Rule* based on the trapezoid rule:
$$\hat{x}(t+h) = x(t) + h\, f(t, x(t)).$$
$$x(t+h) = x(t) + \frac{h}{2}\left[f(t, x(t)) + f(t+h, \hat{x}(t+h))\right],$$
and use the resulting method to solve the differential equation
$$x' = -x + t + \frac{1}{2}$$
$$x(0) = 1$$
over the interval $[0, 1]$ with step size $h = 0.1$.
3. Find the coefficient of h^3 in the error term in the Runge-Kutta method of order 2 by taking the third order Taylor expansion for $x(t+h)$ and comparing it to the third order (multivariable) Taylor expansion for the RK method.
4. Derive the formula for the Runge-Kutta method of order 3 using Mathematica.

   Discussion. In this homework problem, you want to write the Taylor series expansion for $x(t+h)$ and the Taylor series expansion for the RK estimator in terms of f and its partials, and then set the two series equal to one another to solve for the unknown coefficients in the RK estimator. You can expand $x(t+h)$ in a Taylor series using the `Series` command. But how are you going to tell Mathematica to plug in $x'(t) = f(t, x)$?

   Answer. This is an example of a transformation rule. You want Mathematica to replace $x'(t)$ with $f(t, x(t))$, $x''(t)$ with $\frac{d}{dt}f(t, x(t))$ and so forth. This is done with set of rules like

   ```
   D[x[t],{t,1}] -> f[t,x[t]]
   D[x[t],{t,2}] -> D[f[t,x[t]],{t,1}]
   ...
   ```

   But we can actually condense this using Mathematica *patterns*. The transformation rule

   ```
   Sin[x_] -> Cos[x]
   ```

   might be part of a derivative calculator: it causes Mathematica to replace anything in the form `Sin[(something)]` with `Cos[(that thing)]`. Can you use a pattern to reduce the table of rules above to a single rule?

   Comment. The syntax `stuff /. {x -> y}` causes Mathematica to apply the rule `x -> y` **once** to `stuff`. The related command `stuff //. { x -> y }` causes Mathematica to apply `x -> y` to `stuff` until the results stop changing. This is often useful in cases like this one where the application of a rule probably creates another opportunity to use the rule again.
5. Solve the parachute jump problem from the first set of notes numerically using Euler, RK2, RK3, and RK4. Would the parachutist be accelerating or decelerating when he hit? At what speed would he hit? At what height would the answer change from accelerating to decelerating?