Math 4500/6500 Homework #8

This homework assignment covers our notes on Newton-Cotes rules and Gaussian quadrature. You are welcome to look at the code from the *Mathematica* notebooks, but when the problems say "write a piece of code to" they mean "write your own code from scratch", not "modify the code in the notebook" or "find a piece of code on the web". If you find the algebra lengthy or irritating (which is pretty likely), you are encouraged to use *Mathematica* to do it.

1. Consider the integral
$$\int_0^1 \sin\left(\frac{\pi}{2}x^2\right)$$
We wish to integrate this numerically with a relative error no larger than 10^{-3}. What stepsize h is required for the (composite) trapezoid rule? Simpson's rule? Simpson's $\frac{3}{8}$ rule?

2. Use Simpson's rule and its error formula to prove the following surprising result: If a cubic polynomial and a quadratic polynomial agree at three equally spaced points $x-\alpha$, x, and $x+\alpha$, then the two areas enclosed by the graphs of the polynomials are equal.

3. Find a 9-point Newton-Cotes rule using our *Mathematica* code, and test that your formula integrates a polynomial of degree 8 exactly using a particular example.

4. (Challenge + Extra Credit) Implement an adaptive Simpson's rule procedure in *Mathematica*.

5. Evaluate
$$\int_0^2 e^{-x^2}\,dx$$
using Gauss quadrature by making a linear change of variables to change the domain of integration to $[-1,1]$, as required for Gauss quadrature.

6. Verify directly that the formula
$$\int_{-1}^1 f(x)\,dx \sim \frac{5}{9}f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9}f(0) + \frac{5}{9}f\left(\sqrt{\frac{3}{5}}\right).$$
exactly integrates all polynomial $f(x)$ with degree less than or equal to 5. Find a 6th degree polynomial which is not exactly integrated by this formula.

7. Surprisingly, knowing the derivative of a function can be helping in finding the integral. Establish a numerical integration formula of the form
$$\int_a^b f(x)\,dx \sim w_0 f(a) + w_1 f(b) + w_2 f'(a) + w_3 f'(b)$$

which is exact for polynomials of degree as high as possible. Test your integration rule against the four-point Gauss quadrature rule and against Simpson's $\frac{3}{8}$ rule (which also requires four function evaluations) for several non-polynomial functions such as $\cos x$. Does your formula work as well as those methods?

8. (Challenge) The formulae

$$\int_t^{t+h} g(s)\,ds \sim \frac{h}{24}\left[55g(t) - 59g(t-h) + 37g(t-2h) - 9g(t-3h)\right].$$

and

$$\int_t^{t+h} g(s)\,ds \sim \frac{h}{24}\left[9g(t+h) + 19g(t) - 5g(t-h) + g(t-2h)\right].$$

are examples of *Adams-Bashforth-Moulton* integration formulae. Prove that they are exact for polynomials of degree less than or equal to three. Write a composite ABM integrator based on these formulae in *Mathematica* and test it against the composite trapezoid rule and composite Simpson's rule.

9. One of the nice things about the Gaussian quadrature formulae is that the points where the function is evaluated are always interior points of the domain of integration. This means that they can be used to numerically evaluate integrals of functions with singularities at the endpoints of their domains, provided that these improper integrals actually converge. Use Gaussian quadrature to evaluate

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx = \frac{\pi^2}{12}.$$

and see how accurate your results are. Compare your results using Gauss quadrature to your results from several open Newton-Cotes rules, which are the obvious alternative method for singular integrands.