Math 4500/6500 Homework #6

This homework assignment covers our notes on the trapezoid rule and its error analysis. You are welcome to look at the code from the *Mathematica* notebooks, but when the problems say "write a piece of code to" they mean "write your own code from scratch", not "modify the code in the notebook" or "find a piece of code on the web". If you find the algebra lengthy or irritating (which is pretty likely), you are encouraged to use *Mathematica* to do it.

1. Write your own code to compute integrals using the trapezoid rule in *Mathematica*. Use your code to find the result of estimating
$$\int_1^2 \frac{1}{x}\,dx$$
using the trapezoid rule and 3, 5, and 100 points.

2. Prove that if $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then for any sequence of numbers $x_1, \ldots, x_n$ we have
$$\min\{x_i\} \leq \sum_{i=1}^n \lambda_i x_i \leq \max\{x_i\}$$

3. Suppose that $f(x)$ is a concave down function, meaning that the graph of $f(x)$ lies above the chord connecting any two points on the graph. Prove that the trapezoid rule *under*estimates the integral of $f(x)$ over any interval.

4. Find numerical values of each of the three standard integrals below using the *Mathematica* code for the trapezoid rule from problem 1, choosing limits of integration appropriately. Experiment with different limits of integration and numbers of grid points.

   Then make the indicated u-substitution and recompute the values using the trapezoid rule on the new integrals with the same number of grid points. (You don't have to choose the same limits of integration.) Which version of the integral is better integrated by the trapezoid rule? Compare your numerical results to the exact answers given below.

   (1) *The Gaussian Probability Integral*
   $$\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}, \qquad \text{using } x = -\ln t\,.$$

   (2) *The Sinc Integral Formula*
   $$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}, \qquad \text{using } x = \frac{1}{t}\,.$$

   (3) *Fresnel Sine Integral*
   $$\int_0^\infty \sin(x^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}, \qquad \text{using } x = \tan t\,.$$

5. We saw from our discussion of the advanced error analysis of the trapezoid rule that the trapezoid rule should work amazingly well on integrals of periodic functions over a period. Consider the function $f(x) = e^{\sin x}$. Integrate over $[0, 2\pi]$ in *Mathematica* using your trapezoid rule code and compare the results those from *Mathematica*'s built-in `NIntegrate` method. Which is better?