# Math 4500/6500 Homework #5

This homework assignment covers our notes on Richardson Extrapolation and Derivative Calculations. You are welcome to look at the code from the *Mathematica* notebooks, but when the problems say "write a piece of code to" they mean "write your own code from scratch", not "modify the code in the notebook" or "find a piece of code on the web". If you find the algebra lengthy or irritating (which is pretty likely), you are encouraged to use *Mathematica* to do it.

1. Use the Taylor expansion of $f(x)$ to find an expression of the form

$$f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] + E_1$$

for $f'(x)$ where $E_1$ is some error term. This approximation for $f'(x)$ uses two points spaced at distance $2h$, just like the central approximation to the derivative

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + E_2.$$

which we learned in class. Compare the error terms $E_1$ and $E_2$. Which one is better?

2. Use Taylor's Theorem to prove that for any $h > 0$, if $f$ is smooth (has continuous derivatives of every order) on $[x, x+2h]$ then

$$f'(x) - \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)] = \frac{1}{3}h^2 f'''(x) + \text{higher order terms in } h$$

3. Establish the formula

$$f''(x) \sim \frac{2}{h^2}\left[\frac{f(x_0)}{1+\alpha} - \frac{f(x_1)}{\alpha} + \frac{f(x_2)}{\alpha(\alpha+1)}\right]$$

for unequally spaced points $x_0 < x_1 < x_2$ where $x_1 - x_0 = h$ and $x_2 - x_1 = \alpha h$ (instead of $h$) by each of the following two methods:

(1) Approximate $f(x)$ by the Newton form of the interpolating polynomial of degree 2 which interpolates $f$ at $x_0$, $x_1$, and $x_2$ and show that the expression above is the second derivative of that polynomial.

(2) Suppose that the approximation formula was in the general form

$$f''(x) \sim Af(x_0) + Bf(x_1) + Cf(x_2)$$

and solve for the undetermined coefficients by making this formula exact for the three polynomials $1$, $x - x_1$, and $(x - x_1)^2$ (and concluding that the formula is therefore exact for all polynomials of degree $\leq 2$ by linearity).

4. (Challenge) Consider the approximate formula

$$f'(x) \sim \frac{3}{2h^3}\int_{-h}^{h} t f(x+t)\, dt.$$

This is a formula for the *Lanczos Generalized Derivative*.

(1) Prove that if $f$ is differentiable at $x$, the limit of the approximation as $h \to 0$ is $f'(x)$. Hint: L'Hospital's rule.
(2) Find the error term in the expression. (Possibly bogus hint: Try expanding $f(x+t)$ as a Taylor series and integrating term-by-term.)

5. Following the model in the notebook `http://www.jasoncantarella.com/downloads/symbolic_richardson_extrapolation.nb`, use *Mathematica* to find the Richardson extrapolation formulae for the third derivative. Test these formulae for the third derivative of cosine by choosing some point (or points) to evaluate your formulae (with various values of $h$) and comparing your results to the true value of the third derivative of cosine. What is the largest number of correct digits you can get by choosing different values of $h$? How does this compare to the largest number of correct digits you can get for the Richardson extrapolation for the *first* derivative of cosine by choosing $h$ carefully?