Math 4500/6500 Homework #10

This homework assignment covers our notes on the Nelder-Mead algorithm.

1. Code Nelder-Mead on your own in *Mathematica*.

2. The *isoperimetric ratio* of a curve in the plane is the ratio of the area enclosed by the given curve to the square of the length. Notice that this ratio is scale-invariant, and that maximizing the ratio is the same as maximizing the area of the given curve.

   (1) Prove directly that the equilateral triangle has the maximal isoperimetric ratio among all triangles. You might find *Heron's formula* for the area of a triangle in terms of its side lengths to be helpful here.
   (2) Suppose that the first edge of the triangle joins $(0, 0)$ and $(1, 0)$ and the third vertex of the triangle is at (x, y). Write down the isoperimetric ratio as a function $\iota(x, y)$ and use Nelder-Mead in two dimensions to numerically minimize the function $-\iota(x, y)$ in *Mathematica*.
   (3) Create an animation of the search process showing the triangle and its isoperimetric ratio as Nelder-Mead progresses.

3. We can continue the story of the isoperimetric ratio by deriving a general formula for the area enclosed by a polygon joining vertices $v_1 = (x_1, y_1), v_2 = (x_2, y_2), \ldots, v_n = (x_n, y_n)$. We do this using multivariable calculus and the divergence theorem. It's a good idea to spend a few minutes reviewing the divergence theorem, perhaps on Wikipedia, before starting this exercise.

   (1) Verify that the divergence of the vector field $w(x, y) = (x, 0)$ is 1.
   (2) Use the divergence theorem to prove that the area enclosed by any curve γ is equal to the flux of the vector field $w(x, y)$ over γ.
   (3) Evaluate the flux of $w(x, y)$ over the line segment joining (x_k, y_k) and (x_{k+1}, y_{k+1}) in terms of numbers x_k, y_k, x_{k+1}, y_{k+1}.
   (4) Use this flux expression to give a general formula for the area enclosed by vertices $v_1 = (x_1, y_1), \ldots, v_n = (x_n, y_n)$.

4. We continue the story of the isoperimetric ratio as follows.

   (1) Write a *Mathematica* procedure called `Area` using the formula from the last exercise to compute the area enclosed by any list of vertices in the form $\{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$.
   (2) Write a *Mathematica* procedure called `Length` to compute the length of a closed polygon given by a list of vertices in the form $\{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$.
   (3) Combine the two to produce a *Mathematica* procedure `Iso` which computes the isoperimetric ratio.
   (4) Use Nelder-Mead in $2n$ dimensions to find the maximum of the function `Iso` for $n = 3$, $n = 10$, and $n = 100$. Prepare animations of the polygon changing as Nelder-Mead operates.