Math 4250/6250 Homework: Global theory of Curves

This homework assignment covers some global theorems about curves.

1. Obtain a length of wire (pipe cleaner works well) and a ruler (you can print out a ruler if you don't own one). Crumple the wire into a 3 dimensional shape of your choice.

   a. Rotate it and photograph it from at least 6 directions of view. It helps if you stand a little ways away and zoom in, but it's not critical. Make sure that your wire is centered in the image and that the ruler is visible (and square-on to the camera). Submit these photographs with your homework (printouts are fine).

   b. Measure the lengths of the wire in each photograph and record your data. I like to use Fiji `https://imagej.net/Fiji` to do this, but there are plenty of other tools available. Describe your measurement process and submit marked up versions of your images showing how you measured with your homework.

   c. Measure the wire directly using your ruler. Submit a photograph of the wire and the ruler.

   d. Explain how your results agree (or disagree) with the theorem on IntegralGeometric measure for curves.

2. Write an explanation of the Fabricius-Bjerre theorem (not the proof, just the statement) which contains several examples in the form of pictures and is understandable to non-math majors. (This should take a couple of sheets of paper.) Test your explanation on at least 2 non-math majors (this works well with children, if you have any around) and record their reactions.

3. (The change of variables theorem in two dimensions.) Suppose that we have smooth maps $x(u,v) : \mathbf{R}^2 \to \mathbf{R}$ and $y(u,v) : \mathbf{R}^2 \to \mathbf{R}$ from the (u,v)-plane to the (x,y)-plane. Suppose we have a region R in the (u,v) plane and $R^* = \{(x(u,v), y(u,v)) | (u,v) \in R\}$ (that is, R^* is the image of R under the map (x,y)). Recall that

$$\int_{R^*} f(x,y)dxdy = \int_R f(x(u,v), y(u,v)) \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} dudv$$

   a. Consider the explicit functions $x(u,v) = u$, $y(u,v) = v^2$. Write down the relationship between integrating $dxdy$ and $dudv$ using the change of variables formula above.

   b. Let R be the rectangle $[0,2] \times [0,3]$ in the (u,v)-plane and $f(x,y) = x^2 + y^2$. Find R^* in the (x,y) plane and evaluate

$$\int_{R^*} f(x,y)dxdy$$

   Write $g(u,v) = f(x(u,v), y(u,v))$ and compute the equivalent integral

$$\int_R g(u,v) \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} dudv$$

   using the $x(u,v)$ and $y(u,v)$ functions from part a. Make sure that your answers match.

4. We can define a system of coordinates on lines in the plane by

$$\ell(\theta, p) = \{(x, y) | \langle (x, y), (\cos\theta, \sin\theta) \rangle = p\}$$

We say that the "natural measure" on lines in the plane is given by $d\theta dp$. You are now going to prove that this measure doesn't change when we translate or rotate the plane.

   a. Suppose that we map the plane to the plane by the translation $T_{\vec{v}}(x, y) = (x, y) + \vec{v}$. Find the corresponding transformation $t_{\vec{v}}$ of the space of lines. That is, $T_{\vec{v}}(\ell(\theta, p)) = \ell(t_{\vec{v}}(\theta, p))$.

   b. Prove that the transformation on θ, p coordinates given by $(\phi, q) = t_{\vec{v}}(\theta, p)$ has $d\theta dp = d\phi dq$.

   c. Suppose that we map the plane to the plane by R_ψ (rotation around the axis by angle ψ). Find the corresponding transformation r_ψ of the space of lines. That is, $R_\psi(\ell(\theta, p)) = \ell(r_\psi(\theta, p))$.

   d. Prove that the transformation on θ, p coordinates given by $(\phi, q) = r_\psi(\theta, p)$ has $d\theta dp = d\phi dq$.

5. We proved in class that for any plane curve γ

$$\text{Length}(\gamma) = K \int (\#\text{intersections of } \gamma \text{ with } \ell(\theta, p)) d\theta dp.$$

for some constant K. Check the value of K by computing the right-hand integral for a circle of radius r.

6. Suppose we choose a random grid of lines spaced at unit intervals on the plane by choosing θ (uniformly) randomly between 0 and 2π and choosing x (uniformly) randomly between 0 and 1 and letting

$$G(\theta, x) = \{\ell(\theta, k + x) | k \text{ is an integer }\}$$

   a. Compute the expected number of intersections between lines in $G(\theta, x)$ and the line segment S joining $(-L/2, 0)$ and $(L/2, 0)$ by computing the integral

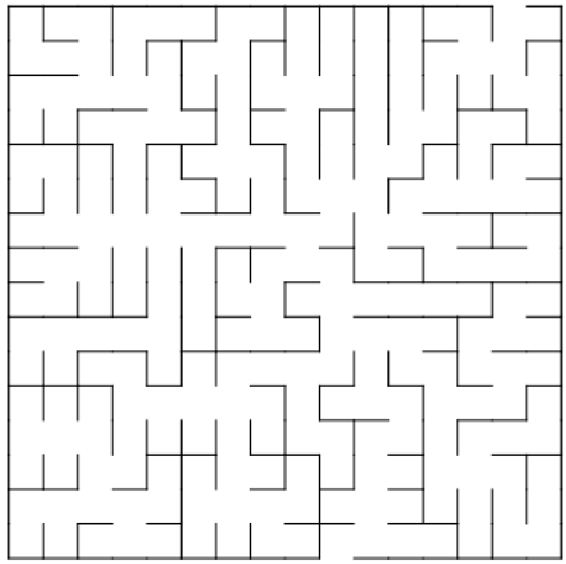
$$\frac{1}{2\pi} \int_0^{2\pi} \int_0^1 (\#\text{intersections of } S \text{ with lines in } G(\theta, x)) d\theta dx.$$

   b. Suppose that γ is any curve of length L in the plane. Compute the expected number of intersections between lines in $G(\theta, x)$ and γ using the results of the last problem.

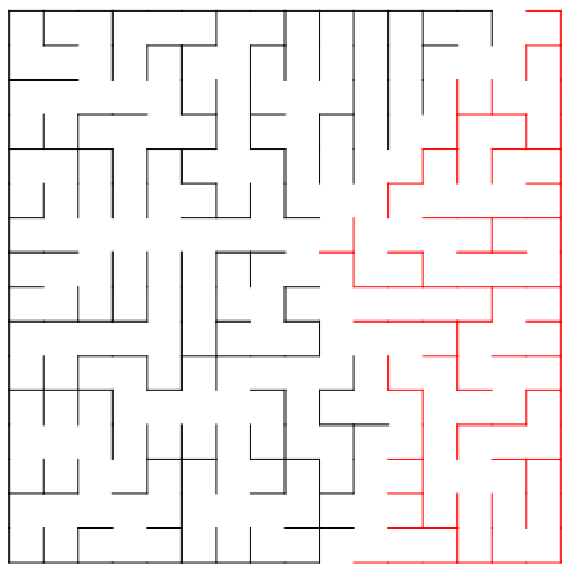
   c. Take a small, easily traceable object (a credit card works pretty well), and throw it randomly onto sheets of lined notebook paper 10 times. Each time, trace the object and count the number of intersections between the outline and the lines on your notebook paper. (Turn in your experiments.) Average this number of intersections and compare it with the perimeter of your object.

7. (A topology problem similar in flavor to Fabricius-Bjerre)

   a. Solve the following maze (turn in your solution):



b. Now start on the left and right sides of an entrance, and color all the walls on one side of the maze black and all walls on the other side red, as below:



c. Suppose that we have a maze M so that the black walls are all connected, the red walls are all connected, and every wall in the maze is either black or red after the coloring is complete. Further, suppose that every square in M touches a wall (perhaps only at corners).

We can define a boundary square of M as one which touches both black and red walls (perhaps only at corners of the square).

Prove that the boundary squares are all connected to each other in a path.

d. Prove that this path solves the maze.

e. Prove that this path is the *shortest* solution to the maze.

f. (Extra credit) Show this trick to a friend or small child and blow their minds.

1. Challenge Problems

1. There's a natural intuition that making a curve of fixed length "three dimensional" involves crumpling the curve and reducing distances between corresponding points (at least on average). This intuition is captured by *Sallee's Stretching Theorem*:

   Theorem 1 (Sallee)**.** *If γ is a closed space curve, there is a corresponding closed plane curve γ^* with the same length so that*

   $$|\gamma(s) - \gamma(t)| \leq |\gamma^*(s) - \gamma^*(t)|$$

   for every pair s, t.

   In this question, you'll prove the theorem. Suppose you have a polygon V in space with vertices $v_1, \ldots, v_n$. Connecting each pair of vertices $v_i v_{i+1}$ to a point p gives a collection of triangles called the "cone of V to p".

   a. The sum of the angles of the vertices of triangles at p is called the intrinsic cone angle $\theta(p)$. Prove that if p is inside the convex hull of V then $\theta(p) \geq 2\pi$.

   b. Now prove that for some p_0, we have $\theta(p_0) = 2\pi$. Hint: Consider the cone angle when p is very far from V.

   c. Now construct a planar polygon V^* by laying the triangles of the cone at p_0 into the plane around the origin. Show that V^* is closed and has the same length as V.

   d. Last, prove the theorem for V and V^*: if both polygons are parametrized by arclength starting at the same vertex v_1, then $|V(s) - V(t)| \geq |V^*(s) - V^*(t)|$ for any s, t. Extend the theorem to smooth curves by approximation.