Math 4250/6250 Homework #2

This homework assignment covers our notes on the Bishop frame (3), Link, Twist, and Writhe (3a) and on the four-vertex theorem. Please pick 5 of the following problems. Remember that undergraduate students should average **one** challenge problem per assignment, while graduate students should average **two** challenge problems per assignment. We'll need the following definitions:

Definition 1. *The line through a point $\vec{p}$ in the direction $\vec{v}$ is parametrized by $\ell(t) = \vec{p} + t\vec{v}$. The* tangent line *to $\alpha(s)$ at s_0 is then $\ell(t) = \alpha(s_0) + tT(s_0)$ (where T is the unit tangent vector) and the* normal line *to $\alpha(s)$ at s_0 is $\ell(t) = \alpha(s_0) + tN(s_0)$.*

We note that you don't want to get confused about s_0 and t here; there is a one-dimensional space of different tangent and normal lines to a given curve parametrized by s *and* a family of points along each individual line parametrized by t. The set of all points on all tangent lines is a surface (called the *tangent developable* surface), but we'll take up that story later in the course.

1. Regular Problems

1. Compute the curvature of the ellipse

$$x = a\cos t, \quad y = b\sin t, \quad t \in [0, 2\pi], \quad a \neq b$$

   and show that it has exactly four vertices (points where the derivative of curvature vanishes). Where are they?

2. Assume that α is a regular space curve (the speed $|\alpha'(t)| \neq 0$) and all the normal lines of the curve (see above) pass though the origin. Prove that α is contained in a circle around the origin.

3. Assume that α is a regular space curve and all of the *tangent* lines of α (see above) pass through the origin. Prove that α is contained in a straight line through the origin. Is this still true if α is not regular?

4. A *translation* is a map $\mathbb{R}^3 \to \mathbb{R}^3$ in the form $A_{\vec{v}}(\vec{p}) = \vec{p} + \vec{v}$ for some fixed vector $\vec{v}$. An *orthogonal linear transformation* is a linear map $B : \mathbb{R}^3 \to \mathbb{R}^3$ so that $\langle B\vec{v}, B\vec{u}\rangle = \langle \vec{v}, \vec{u}\rangle$ for all $\vec{u}, \vec{v} \in \mathbb{R}^3$.
   (1) The group of orthogonal linear transformations is called $O(3)$. The subgroup of orthogonal linear transformations with positive determinant is called $SO(3)$. Prove that the determinant of any $A \in O(3)$ is ± 1 and the determinant of any $A \in SO(3)$ is $+1$.
   (2) Show that for any $A \in SO(3)$ and any vectors $\vec{u}, \vec{v}$, we have

$$|A\vec{u}| = |\vec{u}|,$$

   and the angle θ between $A(\vec{u})$ and $A(\vec{v})$ is the same as the angle between $\vec{u}$ and $\vec{v}$.
   (3) Show that for any $A \in SO(3)$ and any vectors $\vec{u}, \vec{v}$, we have

$$A(\vec{u} \times \vec{v}) = A(\vec{u}) \times A(\vec{v}).$$

   Is this true for all $A \in O(3)$?

(4) A composition of a translation and a map in $SO(3)$ is called a *rigid motion*. Prove that arclength, curvature, and torsion are preserved by rigid motions.

5. Suppose $\alpha : I \to \mathbb{R}^2$ is a regular parametrized plane curve. Assume that the (signed) curvature of α does not vanish. The curve

$$\beta(t) = \alpha(t) + \frac{1}{\kappa(t)} N(t)$$

where N is the Frenet normal is called the *evolute* of α.

(1) Prove that the tangent line of $\beta(t)$ is the normal line of $\alpha(t)$.

(2) Fix some t_0 (it might as well be $t_0 = 0$) and consider the intersection of the normal lines to α at 0 and t, which we will call $\mathrm{nl}_0(u) = \alpha(0) + uN(0)$ and $\mathrm{nl}_t(u) = \alpha(t) + uN(t)$. Let

$$I(0,t) = \text{ the intersection point of } \mathrm{nl}_0 \text{ and } \mathrm{nl}_t\,.$$

Prove that

$$\lim_{t\to 0} I(0,t) = \beta(0).$$

2. Challenge Problems

1. One can give a parametrized plane curve in polar coordinates by specifying $\alpha(\theta) = (r(\theta), \theta)$. In rectangular coordinates, this is the parametrization $\alpha(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$.

(1) Prove that the arclength of α is given by the integral

$$\int \sqrt{r^2 + (r')^2}\, d\theta$$

(2) Prove that the curvature of α is given by the formula

$$\kappa(\theta) = \frac{2(r')^2 - r\,r'' + r^2}{((r')^2 + r^2)^{\frac{3}{2}}}$$

2. Let α be a parametrized regular curve with $\kappa, \tau \neq 0$. The curve is called a *Bertrand curve* if there is another curve β called the *Bertrand mate* of α so that the normal lines to $\alpha(t)$ and $\beta(t)$ are the same line for every t. In this case,

$$\beta(t) = \alpha(t) + r(t)\, N(t)$$

for some scalar function $r(t)$.

(1) Prove that $r(t)$ is a constant function.

(2) Prove that α is a Bertrand curve $\iff$ there are constants A, B so that

$$A\kappa(t) + B\tau(t) = 1$$

(3) Prove if α has more than one Bertrand mate then α is a helix (and hence α has infinitely many Bertrand mates).

3. Prove that if a closed plane curve α is contained inside a disk of radius r, it has a point where the curvature $\kappa > 1/r$. If α is a closed plane curve with length L and total curvature K contained in a disk of radius r, prove the inequality of Istvan Fary:

$$L \leq rK$$

with equality only when α is the circle of radius r.

4. The Bishop frame defines two "curvatures" κ_1 and $\kappa_2(s)$ of a space curve $\alpha(s)$. The curve $(\kappa_1(s), \kappa_2(s))$ is called the "normal development" of the curve α. Compute the Bishop frame and normal development of a helix.

   Hint: Compute the Frenet frame for the helix, and write the Bishop frame $V(s)$ in the form:

   $$V(s) = \cos g(s) N(s) + \sin g(s) B(s),$$

   where $g(s)$ is an unknown function of s. Then use the fact that $V'(s)$ is supposed to be a scalar multiple of the tangent vector $T(s)$ to find a differential equation involving $g'(s)$. Solve that ODE by integration to find $g(s)$, and plug it back into the equation above to find $V(s)$ explicitly.

5. Compute the Gauss linking integral explicitly for the unit circle and the z-axis. (Notice that this is an improper integral over $\mathbf{R} \times S^1$, not $S^1 \times S^1$. But it should still converge.)

6. Suppose a curve $\alpha(s)$ has the property that it's normal lines pass through a fixed point, as in problem #4 on page 23. However, this time do *not* assume that the curve is regular. What can you say about $\alpha(s)$?