Math 4250/6250 Final Exam

This take-home exam covers material from the entire class. Please don't be afraid to read over the notes and the sections in the book as you work on the problems– there is more in the notes than we were able to cover in the lectures, and some of those extra facts might be helpful to you as you work on the exam problems.

As before, please pick 4 of the following problems. You must pick one from each of the four categories: Curve theory, The First Fundamental Form, The Second Fundamental Form, and Geodesics and Gauss-Bonnet. In each category there is an "easier" problem and a "harder" problem. If you are a graduate student, you must attempt **two** "harder" problems. If you are an undergraduate, you will get some bonus credit for doing "harder" problems.

You **are** permitted to use	You **are not** permitted to use
Maple (or Mathematica or MATLAB)	The internet
A calculator (or graphing calculator)	
DoCarmo	Other books
Your notes	Other people's notes
Your brain	Other people's brains
Class notes posted on the website	

The problems are organized by subject area (easier and harder):

1. **(Curve theory, easier)** Suppose we have a strictly convex curve α in the plane (strictly convex means that curvature is everywhere greater than zero). We can parametrize the curve by θ so that the tangent lines to α are all in the form

$$(\cos\theta)x + (\sin\theta)y = p(\theta). \tag{1}$$

   We call $p(\theta)$ the support function of $\alpha(\theta)$. Prove that
   (1) The line in equation (1) is tangent to α at the point

$$\alpha(\theta) = (p(\theta)\cos\theta - p'(\theta)\sin\theta, p(\theta)\sin\theta + p'(\theta)\cos\theta).$$

   (2) The curvature at $\alpha(\theta)$ is given by $1/(p(\theta) + p''(\theta))$.
   (3) The length of α is given by $\int_0^{2\pi} p(\theta)d\theta$.
   (4) **Bonus question.** The area inside α is given by $\frac{1}{2}\int_0^{2\pi}(p(\theta)^2 - p'(\theta)^2)d\theta$.

2. **(Curve theory, harder)** Suppose that we have a plane curve $\alpha(t)$ which is smooth but might not be parametrized by arclength. Prove that if $\|\alpha(s) - \alpha(t)\|$ depends only on $\|s - t\|$ then α is part of a line or a circle.

3. **(The First Fundamental Form, easier)** Gerardus Mercator developed his map projection in 1569. We can think of the Mercator projection as a system of local coordinates on the spherical Earth given by

$$x(u, v) = (\operatorname{sech} u \cos v, \operatorname{sech} u \sin v, \tanh u)$$

   Prove that these are conformal coordinates: angles measured in the u-v plane of the map agree with angles measured on the sphere.

Hint. The following are standard calculus facts about hyperbolic trig functions:

(1) $\operatorname{sech} x = 1/\cosh x$, $\tanh x = \sinh x/\cosh x$.
(2) $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\sinh x = \cosh x$.
(3) $\cosh^2 x - \sinh^2 x = 1$.

4. **(The First Fundamental Form, harder)** The torus of revolution with major radius R and minor radius r is parametrized by

$$x(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$$

There are two obvious families of (round) circles in this surface. Find a third family of circles and draw a picture showing how they fit into the surface. You will almost certainly have to use Maple or Mathematica to complete this problem. (Hint: Look for a plane which is tangent to the torus at exactly **two** points.)

5. **(The Second Fundamental Form, easier)** Prove or give a counterexample:
   (1) If a curve is both an asymptotic curve **and** a line of curvature, then it must be a plane curve.
   (2) If a curve is both an asymptotic curve **and** a plane curve, then it must be a line.

6. **(The Second Fundamental Form, harder)** Suppose that S is a surface with no umbilic points and one principal curvature $k_1 \neq 0$ constant. Prove that M lies on a tube of radius $1/|k_1|$ around a curve. (That is, M is the union of circles of radius $1/|k_1|$ in planes normal to the tangents of a curve α centered at corresponding points on α.)

   Hints: Choose a special parametrization where the coordinate curves u = constant and v = constant are lines of curvature with principal curvatures k_1 and k_2. Use the Mainardi-Codazzi equations to show that the u = constant curves are planar curves of curvature $|k_1|$ (that is, circles). Define α to be the curve formed by the centers of these circles, and check that α is a regular curve.

7. **(Geodesics and Gauss-Bonnet, easier)** Let S be the paraboloid given by $x(u,v) = (u\cos v, u\sin v, u^2)$. Let S_r be the potion of the paraboloid with $0 \leq u \leq r$. Then
   (1) Calculate the geodesic curvature of the boundary of S_r and compute $\int_{\partial S_r} \kappa_g(s)ds$.
   (2) Calculate the Euler characteristic $\chi(S_r)$.
   (3) Use Gauss-Bonnet to compute $\iint_S K d\text{Area}$.
   (4) Compute $\iint_S K d\text{Area}$ explicitly by calculating the curvature of S_r. (Be careful to integrate dArea and not $du\, dv$.)

8. **(Geodesics and Gauss-Bonnet, harder)** The usual tiling of the plane by squares parallel to x and y axes is a tiling by congruent geodesic quadrilaterals so that four such quadrilaterals meet at each vertex of the tiling. Is it possible to tile the sphere with congruent geodesic quadrilaterals in the same way (so that four such quadrangles meet at each vertex)?