Math 4220/6220 Final Exam

This exam covers the entire course. You are allowed to use:

- A linear algebra book of your choice.
- A multivariable calculus book of your choice.
- The first four chapters of Munkres' Topology book.
- MAPLE, Mathematica, Matlab, or any similar calculational aid
- your book
- your notes from class
- the notes posted on the course webpage.

Please don't use any other resources. The exam will be due next Wednesday at 5pm.

During the exam period, please refrain from discussing course material with the other students. You are welcome to ask me questions by email.

Here is an amazing fact, which is useful in several problems:

Theorem 1. *On S^n, every k form ω with $0 < k < n$ and $d\omega = 0$ can be written $\omega = d\alpha$ for some $k-1$ form α. (This follows from the fact that $H^k(S^n; \mathbf{R}) = 0$.)*

Here is another fact from the concluding lecture, which may come in handy:

Theorem 2. *On any compact, orientable manifold X of dimension k, one may define a k-form vol_X so that $\int_X \mathrm{vol}_X = 1$.*

1. Prove or give a counterexample: If X is an orientable manifold, and $N \subset X$ is a submanifold globally cut out by independent functions on X, then N is orientable. Hint: We defined "locally cut out by independent functions" in Chapter 1. What should "globally cut out by independent functions" mean? Make a definition and use it to solve the problem.
2. Suppose that X is a compact, oriented n-dimensional manifold with or without boundary and α is a k-form on X, while ω is an $n-k-1$ form on X. State and prove an "integration by parts" theorem for $\int_X \alpha \wedge d\omega$. Verify that if X is the interval $[a, b]$, this reduces to the standard integration by parts formula. What does the statement say if the manifold X has no boundary?
3. On S^3, suppose we have some 2-form ω with $d\omega = 0$. By the theorem above, $\omega = d\alpha$ for some 1-form α. We can then define the "Helicity" integral
$$H(\omega) = \int_{S^3} \alpha \wedge \omega.$$
   (1) The association of α to ω is certainly not unique (there could be many different α so that $d\alpha = \omega$). Prove that $H(\omega)$ is well-defined.
   (2) Suppose that $f\colon S^3 \to S^3$ is an orientation-preserving diffeomorphism. Prove that $H(\omega) = H(f^*\omega)$ (that is, that H is an orientation-preserving diffeomorphism invariant).

(3) Supose that X is a 3-manifold and we pick some 3-form and call it the volume form: vol_X. Given any vector field V on X, this form allows us to define a corresponding 2-form $\omega(V)$. Given vectors u, w in any tangent space $T_x X$, we let

$$\omega(V)(u, w) = \mathrm{vol}(u, w, V(x)).$$

If Y is a diffeomorphic 3-manifold with a volume form vol_Y, we call a diffeomorphism $f\colon X \to Y$ volume-preserving if

$$f^* \mathrm{vol}_Y = \mathrm{vol}_X .$$

Now while forms pullback under maps, vector fields push forward: given a diffeomorphism $f\colon X \to Y$ and a vector field V on X, we can define a vector field $f_* V$ on Y by $f_* V(y) = df_{f^{-1}(y)}(V(f^{-1}(y)))$.

Using the volume form on Y, we can associate to $f_*(V)$ a 2-form on Y called $\omega(f_*(V))$ as above. If f is orientation preserving and volume-preserving, prove that $\omega(V) = f^*(\omega(f_* V))$.

If $X = Y$ is compact, and $\mathrm{vol}_X = \mathrm{vol}_Y$, prove that it is enough to assume that f is volume-preserving, since any volume-preserving diffeomorphism $f\colon X \to X$ is always orientation preserving. Why?

(4) V is divergence-free if and only if $\omega(V)$ has the property that $d\omega(V) = 0$. (I'd ask you to prove this if we were in $\mathbf{R}^3$, but in S^3 this is pretty much the definition of "divergence-free" for vector fields, so there's no independent way to check this statement.)

For a divergence-free vector field, we can then define $H(V) := H(\omega(V))$. Use the first three results to prove that $H(V)$ is invariant under all volume-preserving diffeomorphisms $S^3 \to S^3$.

4. Suppose X and Y are compact, connected, oriented smooth manifolds without boundary of dimension greater than zero so that $\dim X + \dim Y = k$. Let $f\colon S^k \to X \times Y$ be any smooth map. Use differential forms and our theorem about integrals of pullbacks (and any amazing facts or previous questions that seem helpful) to show that $\deg f = 0$.